UNIVERSIDADE DE SÃO PAULO
Programa de Pos-Graduação em Estatística

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Lista 01 – Semestre de 2024-II – Prof. Gilberto

1. Considere a função densidade de probabilidade da distribuição hiperbólica: $f(y;\theta,\phi) = a(y;\phi)\exp[\phi\{y\theta + (1-\theta^2)^{\frac{1}{2}}\}]$, em que $0 < \theta < 1$, $-\infty < y < \infty$, $\phi^{-1} > 0$ é o parâmetro de dispersão, $a(y;\phi) = \frac{\phi K_1(\phi\sqrt{1+y^2})}{\pi\sqrt{1+y^2}}$ é uma função normalizadora e $K_1(\cdot)$ é a função de Bessel de ordem 1. Mostre que essa distribuição pertence à família exponencial. Encontre a função de variância $V(\mu)$. Obtenha os componentes do desvio $d^{*2}(y_i;\hat{\mu}_i)$ supondo uma amostra de n variáveis aleatórias independentes de médias μ_i e parâmetro de dispersão ϕ^{-1}, para $i = 1,\ldots,n$. Expresse também a medida R^2.

Resolução

(i) Mostrando que a distribuição pertence à família exponencial:

$$
\begin{aligned}
f(y;\theta,\phi) &= a(y;\phi)\exp[\phi\{y\theta + (1-\theta^2)^{\frac{1}{2}}\}]\\
f(y;\theta,\phi) &= \frac{\phi K_1(\phi\sqrt{1+y^2})}{\pi\sqrt{1+y^2}}\exp[\phi\{y\theta + (1-\theta^2)^{\frac{1}{2}}\}]\\
f(y;\theta,\phi) &= \exp\left\{log\left[\frac{\phi K_1(\phi\sqrt{1+y^2})}{\pi\sqrt{1+y^2}}\right]\right\}\exp[\phi\{y\theta + (1-\theta^2)^{\frac{1}{2}}\}]\\
f(y;\theta,\phi) &= \exp\left\{log\left[\frac{\phi K_1(\phi\sqrt{1+y^2})}{\pi\sqrt{1+y^2}}\right] + \phi\left[y\theta + (1-\theta^2)^{\frac{1}{2}}\right]\right\}\\
f(y;\theta,\phi) &= \exp\left\{\phi\left[y\theta - \underbrace{\left\{-(1-\theta^2)^{\frac{1}{2}}\right\}}_{b(\theta)}\right] + \underbrace{\left[log\left(\frac{\phi K_1(\phi\sqrt{1+y^2})}{\pi\sqrt{1+y^2}}\right)\right]}_{c(y;\phi)}\right\}.
\end{aligned}
$$

Como a função densidade de probabilidade tem forma $f(y;\theta,\phi) = \exp\{\phi\,[y\theta - b(\theta)] + c(y;\phi)\}$, esta distribuição pertence à família exponencial (linear).

(ii) Obtendo a média:

Para a família exponencial, vale que

$$\mu = \mathbb{E}(Y) = b'(\theta) = \frac{d}{d\theta}b(\theta).$$

Como $b(\theta) = -(1-\theta^2)^{\frac{1}{2}}$, segue que

$$
\begin{aligned}
\mu &= \frac{d}{d\theta}\left[-(1-\theta^2)^{\frac{1}{2}}\right]\\
\mu &= -\frac{1}{2}(1-\theta^2)^{-\frac{1}{2}}(-2\theta)
\end{aligned}
$$

$$\mu = -\frac{\cancel{1}}{\cancel{2}}(1-\theta^2)^{-\frac{1}{2}}(\cancel{-2}\theta)$$
$$\mu = \theta(1-\theta^2)^{-\frac{1}{2}} = \frac{\theta}{\sqrt{1-\theta^2}}, \quad (0 < \theta < 1).$$

A partir disso, segue que

$$\begin{aligned}
&\mu = \theta\left(1-\theta^2\right)^{-\frac{1}{2}} \Rightarrow \mu^2 = \theta^2\left(1-\theta^2\right)^{-1} \\
&\Rightarrow \frac{1}{\mu^2} = \frac{1-\theta^2}{\theta^2} \Rightarrow \frac{1}{\mu^2} = \frac{1}{\theta^2} - 1 \\
&\Rightarrow \frac{1}{\mu^2} + 1 = \frac{1}{\theta^2} \Rightarrow \frac{1+\mu^2}{\mu^2} = \frac{1}{\theta^2} \\
&\Rightarrow \theta^2 = \frac{\mu^2}{1+\mu^2} \Rightarrow \theta = \left(\frac{\mu^2}{1+\mu^2}\right)^{\frac{1}{2}} \quad (*).
\end{aligned}$$

A última expressão vale uma vez que θ assume valores positivos, ou seja, $0 < \theta < 1$.

(iii) Obtendo a função de variância:

Para a família exponencial, vale que

$$V(\mu) = \frac{d^2b(\theta)}{d\theta^2} = \frac{d}{d\theta}\left[\frac{db(\theta)}{d\theta}\right] = \frac{d\mu}{d\theta}.$$

Então, segue que

$$\begin{aligned}
V(\mu) &= \frac{d}{d\theta}\left(\theta(1-\theta^2)^{-\frac{1}{2}}\right) \\
V(\mu) &= (1-\theta^2)^{-\frac{1}{2}} + \left[-\frac{\cancel{1}}{\cancel{2}}(1-\theta^2)^{-\frac{3}{2}}(\cancel{-2}\cdot\theta)\theta\right] \\
V(\mu) &= (1-\theta^2)^{-\frac{1}{2}} + \theta^2(1-\theta^2)^{-\frac{3}{2}}.
\end{aligned}$$

Utilizando o resultado em (*),

$$\begin{aligned}
V(\mu) &= \left(1-\frac{\mu^2}{1+\mu^2}\right)^{-\frac{1}{2}} + \frac{\mu^2}{1+\mu^2}\left(1-\frac{\mu^2}{1+\mu^2}\right)^{-\frac{3}{2}} \\
V(\mu) &= \left(\frac{1+\mu^2-\mu^2}{1+\mu^2}\right)^{-\frac{1}{2}} + \frac{\mu^2}{1+\mu^2}\left(\frac{1+\mu^2-\mu^2}{1+\mu^2}\right)^{-\frac{3}{2}} \\
V(\mu) &= \left(\frac{1}{1+\mu^2}\right)^{-\frac{1}{2}} + \frac{\mu^2}{1+\mu^2}\left(\frac{1}{1+\mu^2}\right)^{-\frac{3}{2}} \\
V(\mu) &= (1+\mu^2)^{\frac{1}{2}} + \frac{\mu^2}{1+\mu^2}\left(1+\mu^2\right)^{\frac{3}{2}} \\
V(\mu) &= (1+\mu^2)^{\frac{1}{2}} + \frac{\mu^2}{\cancel{1+\mu^2}}\left(1+\mu^2\right)^{\frac{1}{2}}\cancel{(1+\mu^2)} \\
V(\mu) &= (1+\mu^2)^{\frac{1}{2}} + \mu^2\left(1+\mu^2\right)^{\frac{1}{2}} \\
V(\mu) &= (1+\mu^2)^{\frac{1}{2}}(1+\mu^2) \\
V(\mu) &= (1+\mu^2)^{\frac{3}{2}}.
\end{aligned}$$

(iv) Obtendo os componentes do desvio $d^{*2}(y_i;\hat{\mu}_i)$:
Os componentes do desvio são dados por

$$\begin{aligned} d^{*2}(y_i;\hat{\mu}_i) &= \phi d^2(y_i;\hat{\mu}_i) \\ d^{*2}(y_i;\hat{\mu}_i) &= 2\phi\left\{y_i(\tilde{\theta}_i-\hat{\theta}_i)+(b(\hat{\theta}_i)-b(\tilde{\theta}_i))\right\}, i=1,2,\ldots,n. \end{aligned}$$

Segue que

$$b(\theta)=-\left(1-\theta^2\right)^{\frac{1}{2}}=-\left(1-\frac{\mu^2}{1+\mu^2}\right)^{\frac{1}{2}}=-\left(\frac{1+\mu^2-\mu^2}{1+\mu^2}\right)^{\frac{1}{2}}=-\left(\frac{1}{1+\mu^2}\right)^{\frac{1}{2}}=-\frac{1}{\sqrt{1+\mu^2}}.$$

Assim, obtém-se:

$$\begin{aligned} \text{Para o modelo saturado:}\quad \tilde{\theta}_i &= \sqrt{\frac{y_i^2}{1+y_i^2}},\quad b(\tilde{\theta}_i)=-\frac{1}{\sqrt{1+y_i^2}}. \\ \text{Para o modelo sob investigação :}\quad \hat{\theta}_i &= \sqrt{\frac{\hat{\mu}_i^2}{1+\hat{\mu}_i^2}},\quad b(\hat{\theta}_i)=-\frac{1}{\sqrt{1+\hat{\mu}_i^2}}. \end{aligned}$$

Então,

$$\begin{aligned}
d^{*2}(y_i,\hat{\mu}_i) &= \phi d^2(y_i,\hat{\mu}_i) \\
d^{*2}(y_i,\hat{\mu}_i) &= 2\phi\left\{y_i(\tilde{\theta}_i-\hat{\theta}_i)+(b(\hat{\theta}_i)-b(\tilde{\theta}_i))\right\} \\
d^{*2}(y_i,\hat{\mu}_i) &= 2\phi\left\{y_i\left(\sqrt{\frac{y_i^2}{1+y_i^2}}-\sqrt{\frac{\hat{\mu}_i^2}{1+\hat{\mu}_i^2}}\right)+\left[-\frac{1}{\sqrt{1+\hat{\mu}_i^2}}-\left(-\frac{1}{\sqrt{1+y_i^2}}\right)\right]\right\} \\
d^{*2}(y_i,\hat{\mu}_i) &= 2\phi\left\{y_i\left(\frac{y_i}{\sqrt{1+y_i^2}}-\frac{\hat{\mu}_i}{\sqrt{1+\hat{\mu}_i^2}}\right)+\left(-\frac{1}{\sqrt{1+\hat{\mu}_i^2}}+\frac{1}{\sqrt{1+y_i^2}}\right)\right\} \\
d^{*2}(y_i,\hat{\mu}_i) &= 2\phi\left\{y_i\left[\frac{y_i\sqrt{1+y_i^2}}{\left(\sqrt{1+y_i^2}\right)^2}-\frac{\hat{\mu}_i\sqrt{1+\hat{\mu}_i^2}}{\left(\sqrt{1+\hat{\mu}_i^2}\right)^2}\right]+\left[-\frac{\sqrt{1+\hat{\mu}_i^2}}{\left(\sqrt{1+\hat{\mu}_i^2}\right)^2}+\frac{\sqrt{1+y_i^2}}{\left(\sqrt{1+y_i^2}\right)^2}\right]\right\} \\
d^{*2}(y_i,\hat{\mu}_i) &= 2\phi\left\{y_i\left(\frac{y_i\sqrt{1+y_i^2}}{1+y_i^2}-\frac{\hat{\mu}_i\sqrt{1+\hat{\mu}_i^2}}{1+\hat{\mu}_i^2}\right)+\left(-\frac{\sqrt{1+\hat{\mu}_i^2}}{1+\hat{\mu}_i^2}+\frac{\sqrt{1+y_i^2}}{1+y_i^2}\right)\right\} \\
d^{*2}(y_i,\hat{\mu}_i) &= 2\phi\left\{\frac{y_i^2\sqrt{1+y_i^2}}{1+y_i^2}-\frac{\hat{\mu}_iy_i\sqrt{1+\hat{\mu}_i^2}}{1+\hat{\mu}_i^2}+\frac{\sqrt{1+y_i^2}}{1+y_i^2}-\frac{\sqrt{1+\hat{\mu}_i^2}}{1+\hat{\mu}_i^2}\right\} \\
d^{*2}(y_i,\hat{\mu}_i) &= 2\phi\left\{\frac{y_i^2\sqrt{1+y_i^2}}{1+y_i^2}+\frac{\sqrt{1+y_i^2}}{1+y_i^2}-\frac{\hat{\mu}_iy_i\sqrt{1+\hat{\mu}_i^2}}{1+\hat{\mu}_i^2}-\frac{\sqrt{1+\hat{\mu}_i^2}}{1+\hat{\mu}_i^2}\right\} \\
d^{*2}(y_i,\hat{\mu}_i) &= 2\phi\left\{\frac{(y_i^2+1)\sqrt{1+y_i^2}}{1+y_i^2}-\frac{\hat{\mu}_iy_i\sqrt{1+\hat{\mu}_i^2}+\sqrt{1+\hat{\mu}_i^2}}{1+\hat{\mu}_i^2}\right\} \\
d^{*2}(y_i,\hat{\mu}_i) &= 2\phi\left\{\sqrt{1+y_i^2}-\frac{(\hat{\mu}_iy_i+1)\sqrt{1+\hat{\mu}_i^2}}{1+\hat{\mu}_i^2}\right\} \\
d^{*2}(y_i,\hat{\mu}_i) &= 2\phi\left\{\sqrt{1+y_i^2}-\frac{\hat{\mu}_iy_i+1}{\sqrt{1+\hat{\mu}_i^2}}\right\}, i=1,2,\ldots,n.
\end{aligned}$$

(v) Expressão de $\mathbf{R}^2$:

Inicialmente, encontra-se o desvio do modelo, dado por

$$D^*(y;\hat{\mu}) = \sum_{i=1}^{n} d^{*2}(y_i,\hat{\mu}_i)$$
$$D^*(y;\hat{\mu}) = 2\phi\sum_{i=1}^{n}\left\{\sqrt{1+y_i^2} - \frac{\hat{\mu}_i y_i + 1}{\sqrt{1+\hat{\mu}_i^2}}\right\}.$$

Em seguida, encontra-se o desvio do modelo apenas com o intercepto, dado por

$$D^*(y;\bar{y}) = \sum_{i=1}^{n} d^{*2}(y_i,\bar{y})$$
$$D^*(y;\bar{y}) = 2\phi\sum_{i=1}^{n}\left\{\sqrt{1+y_i^2} - \frac{\bar{y} y_i + 1}{\sqrt{1+\bar{y}^2}}\right\}.$$

Então, vale que

$$R^2 = 1 - \frac{D^*(y;\hat{\mu})}{D^*(y;\bar{y})}$$
$$R^2 = 1 - \frac{\cancel{2\phi}\sum_{i=1}^{n}\left\{\sqrt{1+y_i^2} - \frac{\hat{\mu}_i y_i+1}{\sqrt{1+\hat{\mu}_i^2}}\right\}}{\cancel{2\phi}\sum_{i=1}^{n}\left\{\sqrt{1+y_i^2} - \frac{\bar{y} y_i+1}{\sqrt{1+\bar{y}^2}}\right\}}$$
$$R^2 = 1 - \frac{\sum_{i=1}^{n}\left\{\sqrt{1+y_i^2} - \frac{\hat{\mu}_i y_i+1}{\sqrt{1+\hat{\mu}_i^2}}\right\}}{\sum_{i=1}^{n}\left\{\sqrt{1+y_i^2}\right\} - \frac{n\bar{y}^2+n}{\sqrt{1+\bar{y}^2}}}$$
$$R^2 = 1 - \frac{\sum_{i=1}^{n}\left\{\sqrt{1+y_i^2} - \frac{\hat{\mu}_i y_i+1}{\sqrt{1+\hat{\mu}_i^2}}\right\}}{\sum_{i=1}^{n}\left\{\sqrt{1+y_i^2}\right\} - n\sqrt{1+\bar{y}^2}}.$$

2. Supor que $Y_1,\ldots,Y_k$ sejam variáveis aleatórias independentes tais que Y_i tenha função de probabilidade expressa na forma

$$f(y_i;\psi) = \binom{m}{y_i}\left(\frac{\psi}{1+\psi}\right)^{y_i}\left(\frac{1}{1+\psi}\right)^{m-y_i},$$

com $\psi > 0$ e $y_i = 0,1,\ldots,m$ e $i = 1,\ldots,k$. Como fica a estatística do teste da razão de verossimilhanças para testar $H_0 : \psi = 1$ contra $H_1 : \psi \neq 1$? Qual a distribuição nula assintótica da estatística do teste?

Resolução

$$f(y_i;\psi) = \binom{m}{y_i}\left(\frac{\psi}{1+\psi}\right)^{y_i}\left(\frac{1}{1+\psi}\right)^{m-y_i}, i = 1,\ldots,k,$$

Como as variáveis aleatórias são independentes e identicamente distribuídas, a função de verossimilhança é dada por

$$L(\psi;y_1,\ldots,y_k) \stackrel{iid}{=} \prod_{i=1}^{k}\binom{m}{y_i}\left(\frac{\psi}{1+\psi}\right)^{y_i}\left(\frac{1}{1+\psi}\right)^{m-y_i}$$

A log-verossimilhança é então dada por

$$
\begin{aligned}
\ell(\psi) &= \log\left[\prod_{i=1}^{k}\binom{m}{y_i}\left(\frac{\psi}{1+\psi}\right)^{y_i}\left(\frac{1}{1+\psi}\right)^{m-y_i}\right]\\
\ell(\psi) &= \sum_{i=1}^{k}\left[\log\left(\binom{m}{y_i}\left(\frac{\psi}{1+\psi}\right)^{y_i}\left(\frac{1}{1+\psi}\right)^{m-y_i}\right)\right]\\
\ell(\psi) &= \sum_{i=1}^{k}\left[\log\binom{m}{y_i}+y_i\log\left(\frac{\psi}{1+\psi}\right)+(m-y_i)\log\left(\frac{1}{1+\psi}\right)\right]\\
\ell(\psi) &= \sum_{i=1}^{k}\left[\log\binom{m}{y_i}+y_i\log(\psi)-y_i\log(1+\psi)-(m-y_i)\log(1+\psi)\right]\\
\ell(\psi) &= \sum_{i=1}^{k}\left[\log\binom{m}{y_i}+y_i\log(\psi)-y_i\log(1+\psi)-m\log(1+\psi)+y_i\log(1+\psi)\right]\\
\ell(\psi) &= \sum_{i=1}^{k}\left[\log\binom{m}{y_i}+y_i\log(\psi)-m\log(1+\psi)\right]\\
&\Rightarrow \boxed{\ell(\psi) = \sum_{i=1}^{k}\log\binom{m}{y_i}+\log(\psi)\sum_{i=1}^{k}y_i-mk\log(1+\psi)}.
\end{aligned}
$$

Segue então que

$$
\frac{d\ell(\psi)}{d\psi} = \frac{1}{\psi}\sum_{i=1}^{k}y_i-\frac{mk}{1+\psi} \quad \text{e} \quad \frac{d^2\ell(\psi)}{d\psi^2} = -\frac{1}{\psi^2}\sum_{i=1}^{k}y_i+\frac{mk}{(1+\psi)^2}.
$$

Quando $\frac{d\ell(\psi)}{d\psi}=0$,

$$
\begin{aligned}
&\frac{1}{\hat\psi}\sum_{i=1}^{k}y_i-\frac{mk}{1+\hat\psi}=0 \iff \frac{1}{\hat\psi}\sum_{i=1}^{k}y_i=\frac{mk}{1+\hat\psi} \iff \frac{1}{\hat\psi}(1+\hat\psi)=\frac{mk}{\sum_{i=1}^{k}y_i} \iff \frac{1}{\hat\psi}+1=\frac{mk}{\sum_{i=1}^{k}y_i}\\
&\iff \frac{1}{\hat\psi}=\frac{mk}{\sum_{i=1}^{k}y_i}-1 \iff \frac{1}{\hat\psi}=\frac{mk-\sum_{i=1}^{k}y_i}{\sum_{i=1}^{k}y_i} \iff \hat\psi=\frac{\sum_{i=1}^{k}y_i}{mk-\sum_{i=1}^{k}y_i}\\
&\hat\psi=\frac{\sum_{i=1}^{k}y_i}{mk-\sum_{i=1}^{k}y_i}\times\frac{\frac{1}{k}}{\frac{1}{k}}\\
&\hat\psi=\frac{\bar y}{m-\bar y},\quad \bar y\neq m.
\end{aligned}
$$

Em $\hat\psi=\frac{\bar y}{m-\bar y}$,

$$
\begin{aligned}
\frac{d^2\ell(\hat\psi)}{d\psi^2} &= -\frac{1}{(\frac{\bar y}{m-\bar y})^2}\sum_{i=1}^{k}y_i+\frac{mk}{(1+\frac{\bar y}{m-\bar y})^2}\\
\frac{d^2\ell(\hat\psi)}{d\psi^2} &= -\frac{(m-\bar y)^2}{\bar y^{\cancel{2}}}k\cancel{\bar y}+\frac{mk}{(\frac{m}{m-\bar y})^2}\\
\frac{d^2\ell(\hat\psi)}{d\psi^2} &= -\frac{k(m-\bar y)^2}{\bar y}+\frac{\cancel{m}k(m-\bar y)^2}{m^{\cancel{2}}}\\
\frac{d^2\ell(\hat\psi)}{d\psi^2} &= -\frac{k(m-\bar y)^2}{\bar y}+\frac{k(m-\bar y)^2}{m}<0,
\end{aligned}
$$

uma vez que $m > \bar{y}$. Logo, $\hat{\psi} = \frac{\bar{y}}{m-\bar{y}}$ maximiza a função de verossimilhança para $\bar{y} \neq m$.
A estatística do teste da razão de verossimilhanças é dada por

$$\xi_{RV} = 2\left(\ell(\hat{\psi}) - \ell(\psi_0)\right),$$

onde $\psi_0 = 1$ sob a hipótese nula $H_0 : \psi = 1$.
Substituindo $\psi_0 = 1$ na log-verossimilhança:

$$\ell(\psi_0 = 1) = \sum_{i=1}^{k} \log\binom{m}{y_i} + \log(1)\sum_{i=1}^{k} y_i - mk\log(1+1) = \boxed{\sum_{i=1}^{k} \log\binom{m}{y_i} - mk\log(2)}, \quad \log(1) = 0.$$

Para $\hat{\psi}$:

$$\ell(\hat{\psi}) = \sum_{i=1}^{k} \log\binom{m}{y_i} + \log(\hat{\psi})\sum_{i=1}^{k} y_i - mk\log(1+\hat{\psi})$$
$$\ell(\hat{\psi}) = \sum_{i=1}^{k} \log\binom{m}{y_i} + \log\left(\frac{\bar{y}}{m-\bar{y}}\right)\sum_{i=1}^{k} y_i - mk\log\left(1+\frac{\bar{y}}{m-\bar{y}}\right)$$
$$\Rightarrow \boxed{\ell(\hat{\psi}) = \sum_{i=1}^{k} \log\binom{m}{y_i} + \log\left(\frac{\bar{y}}{m-\bar{y}}\right)\sum_{i=1}^{k} y_i - mk\log\left(\frac{m}{m-\bar{y}}\right)}.$$

Logo, a estatística do teste é:

$$\xi_{RV} = 2\left(\ell(\hat{\psi}) - \ell(\psi_0)\right)$$
$$\xi_{RV} = 2\left[\sum_{i=1}^{k} \log\binom{m}{y_i} + \log\left(\frac{\bar{y}}{m-\bar{y}}\right)\sum_{i=1}^{k} y_i - mk\log\left(\frac{m}{m-\bar{y}}\right) - \left(\sum_{i=1}^{k} \log\binom{m}{y_i} - mk\log(2)\right)\right]$$
$$\xi_{RV} = 2\left[\sum_{i=1}^{k} \log\binom{m}{y_i} + \log\left(\frac{\bar{y}}{m-\bar{y}}\right)\sum_{i=1}^{k} y_i - mk\log\left(\frac{m}{m-\bar{y}}\right) - \sum_{i=1}^{k} \log\binom{m}{y_i} + mk\log(2)\right]$$
$$\xi_{RV} = 2\left[\log\left(\frac{\bar{y}}{m-\bar{y}}\right)\sum_{i=1}^{k} y_i + mk\left[\log(2) - \log\left(\frac{m}{m-\bar{y}}\right)\right]\right]$$
$$\xi_{RV} = 2\left[k\bar{y}\log\left(\frac{\bar{y}}{m-\bar{y}}\right) + mk\log\left(\frac{2(m-\bar{y})}{m}\right)\right]$$
$$\xi_{RV} \underset{k\to\infty}{\overset{H_0}{\sim}} \chi^2_{(1)}$$

Sob a hipótese nula $H_0 : \psi = 1$, quando o tamanho da amostra tende ao infinito ($k \to \infty$), a estatística do teste da razão de verossimilhanças segue, assintoticamente, uma distribuição qui-quadrado com 1 grau de liberdade ($\chi^2_{(1)}$), pois estamos testando um único parâmetro (ψ).

3. Supor que $Z_i \overset{iid}{\sim} ZAP(\mu, \pi)$, para $i = 1, \ldots, n$, em que a função de probabilidades de Z_i é dada por

$$f_Z(z_i; \mu, \pi) = \begin{cases} \pi & \text{se } z_i = 0, \\ (1-\pi)\frac{f_Y(z_i;\mu)}{1-f_Y(0;\mu)} & \text{se } z_i = 1, 2, \ldots, \end{cases}$$

em que $f_Y(y_i; \mu)$ denota a função de probabilidades de uma $P(\mu)$. Obter a estatística do teste da razão de verossimilhanças para testar $H : \mu = 1$ contra $A : \mu \neq 1$. Qual é a distribuição nula assintótica da estatística do teste?

Resolução

$$f_Z(z_i;\mu,\pi) = \pi I_{\{0\}}(z_i) + (1-\pi)\frac{f_Y(z_i;\mu)}{1-f_Y(0;\mu)} I_{\{\mathbb{N}^*\}}(z_i)$$

Como as variáveis aleatórias são independentes e identicamente distribuídas, a função de verossimilhança é dada por

$$L(\mu,\pi;z_1,\ldots,z_k) \stackrel{iid}{=} \prod_{i=1}^{n}\left[\pi I_{\{0\}}(z_i) + (1-\pi)\frac{f_Y(z_i;\mu)}{1-f_Y(0;\mu)} I_{\{\mathbb{N}^*\}}(z_i)\right].$$

Suponha que as r primeiras observações são zero

$$\begin{aligned}
L(\mu,\pi) &= \prod_{i=1}^{r}\pi \prod_{i=r+1}^{n}\left[(1-\pi)\frac{f_Y(z_i;\mu)}{1-f_Y(0;\mu)}\right]\\
\ell(\mu,\pi) &= \log\left(\prod_{i=1}^{r}\pi\right) + \log\left(\prod_{i=r+1}^{n}\left[(1-\pi)\frac{f_Y(z_i;\mu)}{1-f_Y(0;\mu)}\right]\right)\\
\ell(\mu,\pi) &= \log(\pi^r) + \sum_{i=r+1}^{n}\log\left[(1-\pi)\frac{f_Y(z_i;\mu)}{1-f_Y(0;\mu)}\right], \quad \text{observe que, } \sum_{i=r+1}^{n} = \sum_{i=1}^{n-r}\\
\ell(\mu,\pi) &= r\log(\pi) + \sum_{i=1}^{n-r}\left(\log(1-\pi) + \log\left[f_Y(z_i;\mu)\right] - \log\left[1-f_Y(0;\mu)\right]\right)\\
\ell(\mu,\pi) &= r\log(\pi) + \sum_{i=1}^{n-r}\log(1-\pi) + \sum_{i=1}^{n-r}\log\left[f_Y(z_i;\mu)\right] - \sum_{i=1}^{n-r}\log\left[1-f_Y(0;\mu)\right]
\end{aligned}$$

Observe que $Y \sim Pois(\mu)$:$f_Y(z_i;\mu) = \frac{\mu^{z-i}e^{-\mu}}{z_i!}$ e se $z_i = 0$ tem-se $f_Y(0;\mu) = e^{-\mu}$

$$\ell(\mu,\pi) = r\log(\pi) + (n-r)\log(1-\pi) + \sum_{i=1}^{n-r}\log\left(\frac{\mu^{z_i}e^{-\mu}}{z_i!}\right) - \sum_{i=1}^{n-r}\log\left(1-e^{-\mu}\right)$$

$$\ell(\mu,\pi) = r\log(\pi) + (n-r)\log(1-\pi) + \sum_{i=1}^{n-r}\left[\log(\mu^{z_i}) + \log\left(e^{-\mu}\right) - \log(z_i!)\right] - (n-r)\log\left(1-e^{-\mu}\right)$$

$$\ell(\mu,\pi) = r\log(\pi) + (n-r)\log(1-\pi) + \sum_{i=1}^{n-r} z_i\log(\mu) - \sum_{i=1}^{n-r}\mu - \sum_{i=1}^{n-r}\log(z_i!) - (n-r)\log\left(1-e^{-\mu}\right)$$

$$\ell(\mu,\pi) = r\log(\pi) + (n-r)\log(1-\pi) + \log(\mu)\sum_{i=1}^{n-r} z_i - (n-r)\mu - \sum_{i=1}^{n-r}\log(z_i!) - (n-r)\log\left(1-e^{-\mu}\right)$$

Note que como $z_i = 0$ para $i = 1,\cdots,r$, $\sum_{i=1}^{n-r} z_i = \sum_{i=1}^{n} z_i$ e $\sum_{i=1}^{n-r}\log(z_i!) = \sum_{i=1}^{n}\log(z_i!)$.

$$\ell(\mu,\pi) = r\log(\pi) + (n-r)\log(1-\pi) + n\log(\mu)\,\bar{z} - (n-r)\mu - \sum_{i=1}^{n}\log(z_i!) - (n-r)\log\left(1-e^{-\mu}\right)$$

$$\frac{d\ell(\mu,\pi)}{d\mu} = \frac{n}{\mu}\bar{z} - (n-1) - \frac{(n-r)e^{-\mu}}{1-e^{-\mu}} = \boxed{\frac{1}{\mu}n\bar{z} - (n-r) - \frac{n-r}{e^{\mu}-1}}$$

Quando $\frac{d\ell(\mu,\pi)}{d\mu} = 0$,

$\dfrac{1}{\hat{\mu}}n\bar{z} - (n-r) - \dfrac{n-r}{e^{\hat{\mu}}-1} = 0,$ não pode ser resolvido analiticamente, portanto utilizamos o método de Newton-Raphson para encontrar numericamente $\hat{\mu}$.

$$\text{Seja } g(\hat{\mu}) = \frac{n}{\hat{\mu}}\bar{z} - \frac{n-r}{e^{\hat{\mu}}-1} - (n-r)$$

$$G(\hat{\mu}) = \frac{dg(\hat{\mu})}{d\hat{\mu}} = -\frac{n}{\hat{\mu}^2}\bar{z} - \frac{(n-r)e^{\hat{\mu}}}{(e^{\hat{\mu}}-1)^2}$$

$$\hat{\mu}^{(m+1)} = \hat{\mu}^{(m)} - \frac{g(\hat{\mu}^m)}{G(\hat{\mu}^{(m)})}, \quad m = 0, 1, 2, 3, \ldots$$

$$\hat{\mu}^{(m+1)} = \hat{\mu}^{(m)} - \frac{g(\hat{\mu}^m)}{G(\hat{\mu}^{(m)})}, \quad m = 0, 1, 2, 3, \ldots$$

$$\hat{\mu}^{(m+1)} = \hat{\mu}^{(m)} - \frac{\frac{n}{\hat{\mu}^{(m)}}\bar{z} - \frac{(n-r)}{e^{\hat{\mu}^{(m)}}-1} - (n-r)}{-\frac{n}{(\hat{\mu}^{(m)})^2}\bar{z} + \frac{(n-r)e^{\hat{\mu}^{(m)}}}{(e^{\hat{\mu}^{(m)}}-1)^2}}, \quad m = 0, 1, 2, 3, \ldots$$

O sobrescrito (m) indica que as expressões são avaliadas no valor de μ na iteração m. Repetimos o procedimento até que o seguinte critério de convergência seja satisfeito:

$$\left| \frac{\hat{\mu}^{(m+1)} - \hat{\mu}^{(m)}}{\hat{\mu}^{(m)}} \right| < \epsilon,$$

para algum $\epsilon > 0$ suficientemente pequeno.

É importante destacar que procedimentos iterativos tais como o de Newton Raphson (NR) requerem um chute inicial. Nesse caso, sugerimos utilizar a média amostral como valor inicial para a estimativa de μ, ou seja,

$$\hat{\mu}^{(0)} = \frac{1}{n}\sum_{i=1}^{n} z_i.$$

Seja $\hat{\mu}$ o estimador de máxima verossimilhança encontrado em ou após $m+1$ iterações, em que se alcança a convergência.

Razão de Verossimilhanças

A estatística do teste de verossimilhança é dada por

$$\xi_{RV} = 2\{\ell(\hat{\mu}, \hat{\pi}) - \ell(\mu^0, \hat{\pi}^0)\}$$

Para acharmos $\hat{\pi}$ e $\hat{\pi}^0$ segue que

$$\ell(\mu, \pi) = r\log(\pi) + (n-r)\log(1-\pi) + n\log(\mu)\,\bar{z} - (n-r)\mu - \sum_{i=1}^{n}\log(z_i!) - (n-r)\log\left(1 - e^{-\mu}\right)$$

$$\frac{d\ell(\mu,\pi)}{d\pi} = 0 \iff \frac{r}{\hat{\pi}} - \frac{n-r}{1-\hat{\pi}} = 0 \iff \frac{r}{\hat{\pi}} = \frac{n-r}{1-\hat{\pi}} \iff \frac{1-\hat{\pi}}{\hat{\pi}} = \frac{n-r}{r} \iff \frac{1}{\hat{\pi}} - 1 = \frac{n-r}{r}$$

$$\frac{1}{\hat{\pi}} = \frac{n-r}{r} + 1 = \frac{n}{r}$$

$$\Rightarrow \boxed{\hat{\pi} = \frac{r}{n},\ n \in \mathbb{N}^*}, \text{ observe que } \hat{\pi} \text{ não depende de } \mu, \text{ logo, } \hat{\pi}^0 = \hat{\pi}.$$

Segue então que

$$\Rightarrow \boxed{\ell(\hat{\mu}, \hat{\pi}) = r\log(\hat{\pi}) + (n-r)\log(1-\hat{\pi}) + n\log(\hat{\mu})\,\bar{z} - (n-r)\hat{\mu} - \sum_{i=1}^{n}\log(z_i!) - (n-r)\log\left(1 - e^{-\hat{\mu}}\right)}.$$

Também,

$$\ell(\mu^0,\hat{\pi}^0) = r\log(\hat{\pi}) + (n-r)\log(1-\hat{\pi}) + n\log\left(\mu^0\right)\bar{z} - (n-r)\mu^0 - \sum_{i=1}^{n}\log(z_i!) - (n-r)\log\left(1-e^{-\mu^0}\right)$$

$$\ell(\mu^0,\hat{\pi}^0) = r\log(\hat{\pi}) + (n-r)\log(1-\hat{\pi}) + n\log(1)\bar{z} - (n-r)\cdot 1 - \sum_{i=1}^{n}\log(z_i!) - (n-r)\log\left(1-e^{-1}\right)$$

$$\Longrightarrow \boxed{\ell(\mu^0,\hat{\pi}^0) = r\log(\hat{\pi}) + (n-r)\log(1-\hat{\pi}) - (n-r) - \sum_{i=1}^{n}\log(z_i!) - (n-r)\log\left(1-e^{-1}\right)}$$

$$\xi_{RV} = 2\{\ell(\hat{\mu},\hat{\pi}) - \ell(\mu^0,\hat{\pi}^0)\}$$

$$\xi_{RV} = 2\left\{r\log(\hat{\pi}) + (n-r)\log(1-\hat{\pi}) + n\log(\hat{\mu})\bar{z} - (n-r)\hat{\mu} - \sum_{i=1}^{n}\log(z_i!) - (n-r)\log\left(1-e^{-\hat{\mu}}\right)\right.$$
$$\left. - \left[r\log(\hat{\pi}) + (n-r)\log(1-\hat{\pi}) - (n-r) - \sum_{i=1}^{n}\log(z_i!) - (n-r)\log\left(1-e^{-1}\right)\right]\right\}$$

$$\xi_{RV} = 2\left\{r\log(\hat{\pi}) + (n-r)\log(1-\hat{\pi}) + n\log(\hat{\mu})\bar{z} - (n-r)\hat{\mu} - \sum_{i=1}^{n}\log(z_i!) - (n-r)\log\left(1-e^{-\hat{\mu}}\right)\right.$$
$$\left. -r\log(\hat{\pi}) - (n-r)\log(1-\hat{\pi}) + (n-r) + \sum_{i=1}^{n}\log(z_i!) + (n-r)\log\left(1-e^{-1}\right)\right\}$$

$$\xi_{RV} = 2\left\{n\log(\hat{\mu})\bar{z} - (n-r)\hat{\mu} - (n-r)\log\left(1-e^{-\hat{\mu}}\right) + (n-r) + (n-r)\log\left(1-e^{-1}\right)\right\}$$

$$\Longrightarrow \xi_{RV} \underset{n\longrightarrow\infty}{\overset{H_0}{\sim}} \chi^2_{(1)},$$

ou seja, sob H_0, quando o número de amostra tende ao infinito ($n \longrightarrow \infty$), a estatística do teste da razão de verossimilhanças segue, assintoticamente, uma distribuição qui-quadrado com graus de liberdade ($g.l.$) iguais à diferença no número de parâmetros livres entre os dois modelos comparados. No modelo sob H_0, temos 1 parâmetro livre (π). Já no modelo sob H_A, temos 2 parâmetros livres (μ e π). Portanto, a diferença no número de parâmetros é $2-1=1$, resultando em 1 grau de liberdade para a distribuição da estatística do teste.

4. Supor que $Z \sim \text{ZANBI}(\mu,\nu,\pi)$ e $U \sim \text{ZINBI}(\mu,\nu,\pi)$. Mostrar que ambas as distribuições são funções de probabilidade (fp). Obter $\mathbb{E}(Z)$, $Var(Z)$, $\mathbb{E}(U)$ e $Var(U)$ deixando essas quantidades em função dos parâmetros (μ,ν,π).

Resolução

$\Longrightarrow$ Se $Z\sim$ ZANBI (μ, ν, π),

$$f_z(z_i; \mu, \nu, \pi) = \begin{cases} \pi & \text{se } z_i = 0 \\ (1-\pi)\frac{f_y(z_i;\mu,\nu)}{1-f_y(0;\mu,\nu)} & \text{se } z_i = 1, 2, \ldots, \end{cases}$$

em que $f_y(z_i; \mu, \nu)$ denota a função de probabilidade de uma BN (μ, ν).

$$f_y(z; \mu, \nu) = \frac{\Gamma(z+\nu)}{\Gamma(\nu)\Gamma(z+1)}\left(\frac{\mu}{\mu+\nu}\right)^z\left(\frac{\nu}{\mu+\nu}\right)^\nu$$

$$f_y(0; \mu, \nu) = \frac{\Gamma(0+\nu)}{\Gamma(\nu)\Gamma(1)}\left(\frac{\mu}{\mu+\nu}\right)^0\left(\frac{\nu}{\mu+\nu}\right)^\nu = \left(\frac{\nu}{\mu+\nu}\right)^\nu, \quad \Gamma(n) = (n-1)!,\ 0! = 1$$

$\Longrightarrow$Para ser fp devem satisfazer: 1)$f_z(z_i; \mu, \nu, \pi) \geq 0$ e 2) $\sum_{-\infty}^{+\infty} f_z(z_i; \mu, \nu, \pi) = 1$

$\Longrightarrow$ Condição 1)

$$\sum_{-\infty}^{+\infty} f_y(z_i; \mu, \nu) = f_y(0; \mu, \nu) + \sum_{i=1}^{+\infty} f_y(z_i; \mu, \nu) = 1 \Longrightarrow 0 \leq f_y(0; \mu, \nu) \leq 1 \ \text{ e } 0 \leq 1 - f_y(0; \mu, \nu) \leq 1$$

$$\sum_{-\infty}^{+\infty} f_z(z_i; \mu, \nu, \pi) = f_z(0; \mu, \nu, \pi) + \sum_{i=1}^{+\infty} f_z(z_i; \mu, \nu, \pi) = \pi + \sum_{i=1}^{+\infty} f_z(z_i; \mu, \nu, \pi) = 1$$

$$\sum_{i=1}^{+\infty} f_z(z_i; \mu, \nu, \pi) = 1 - \pi \Longleftrightarrow \boxed{0 \leq 1 - \pi \leq 1}$$

$\boxed{f_y(z_i; \mu, \nu) \geq 0}$, porque é fp

Pelas desigualidades apresentadas acima, se nota que:

$$0 \leq (1-\pi)\frac{f_y(z_i; \mu, \nu)}{1 - f_y(0; \mu, \nu)} \leq 1, \text{ e como consequência } \boxed{f_z(z_i; \mu, \nu, \pi) \geq 0}$$

$\Longrightarrow$ Condição 2)

$$\sum_{-\infty}^{+\infty} f_z(z_i; \mu, \nu, \pi) = 1$$

$$\sum_{-\infty}^{+\infty} f_z(z_i; \mu, \nu, \pi) = f_z(0; \mu, \nu, \pi) + \sum_{z_i=1}^{+\infty} f_z(z_i; \mu, \nu, \pi)$$

$$\sum_{-\infty}^{+\infty} f_z(z_i; \mu, \nu, \pi) = \pi + \frac{(1-\pi)}{1 - f_y(0; \mu, \nu)}\sum_{z_i=1}^{+\infty} f_y(z_i; \mu, \nu)$$

$$\sum_{-\infty}^{+\infty} f_z(z_i; \mu, \nu, \pi) = \pi + \frac{(1-\pi)}{1 - f_y(0; \mu, \nu)}(1 - f_y(0; \mu, \nu)), \text{ porque} \sum_{z_i=0}^{+\infty} f_y(z_i; \mu, \nu) = f_y(0; \mu, \nu) + \sum_{z_i=1}^{+\infty} f_y(z_i; \mu, \nu) = 1$$

$$\sum_{-\infty}^{+\infty} f_z(z_i; \mu, \nu, \pi) = \pi + 1 - \pi$$

$$\sum_{-\infty}^{+\infty} f_z(z_i; \mu, \nu, \pi) = 1$$

Logo, como as duas condições foram satisfeitas,

$$f_z(z_i; \mu, \nu, \pi) = \begin{cases} \pi & \text{se } z_i = 0 \\ (1-\pi)\frac{\frac{\Gamma(z+\nu)}{\Gamma(\nu)\Gamma(z+1)}\left(\frac{\mu}{\mu+\nu}\right)^z\left(\frac{\nu}{\mu+\nu}\right)^\nu}{1-\left(\frac{\nu}{\mu+\nu}\right)^\nu} & \text{se } z_i = 1, 2, \ldots, \end{cases}$$

é fp.

$\Longrightarrow$ Esperança de Z

$$\mathbb{E}(Z) = \sum_{z_i=0}^{+\infty} z f_z(z_i;\mu,\nu,\pi) = \sum_{z_i=0}^{+\infty} z f_z(z_i;\mu,\nu,\pi) = 0\cdot f_z(0;\mu,\nu,\pi) + \sum_{z_i=1}^{+\infty} z f_z(z_i;\mu,\nu,\pi)$$

$$\mathbb{E}(Z) = \sum_{z_i=1}^{+\infty} z f_z(z_i;\mu,\nu,\pi) = \frac{(1-\pi)}{1-f_y(0;\mu,\nu)}\sum_{z_i=1}^{+\infty} z f_y(z_i;\mu,\nu) = \frac{(1-\pi)}{1-f_y(0;\mu,\nu)}\left[\mathbb{E}(Y) - 0\cdot f_y(0;\mu,\nu)\right], \quad (**)$$

$$\mathbb{E}(Z) = \frac{(1-\pi)}{1-f_y(0;\mu,\nu)}\mathbb{E}(Y)$$

$$f_y(z;\mu,\nu) = \frac{\Gamma(z+\nu)}{\Gamma(\nu)\Gamma(z+1)}\left(\frac{\mu}{\mu+\nu}\right)^z\left(\frac{\nu}{\mu+\nu}\right)^\nu, \quad Y\sim BN(\mu,\nu)$$

$$f_y(z;\mu,\nu) = \exp\left\{z\log\left(\frac{\mu}{\mu+\nu}\right) + \log\left(\frac{\nu}{\mu+\nu}\right)^\nu + \log\left(\frac{\Gamma(z+\nu)}{\Gamma(\nu)\Gamma(z+1)}\right)\right\}, \text{F.E de } \phi = 1$$

$$\begin{cases} b(\theta) & = -\log\left(\frac{\nu}{\mu+\nu}\right)^\nu \\ \theta & = \log\left(\frac{\mu}{\mu+\nu}\right)\end{cases} \iff \begin{cases} b(\theta) & = \log\left(\frac{\mu+\nu}{\nu}\right)^\nu \\ e^\theta & = \frac{\mu}{\mu+\nu}\end{cases} \iff \begin{cases} b(\theta) & = \nu\log\left(1+\frac{\mu}{\nu}\right) \\ \frac{1}{e^\theta} & = 1+\frac{\nu}{\mu}\end{cases} \underset{\dots}{\iff} \begin{cases} b(\theta) & = \nu\log\left(1+\frac{\mu}{\nu}\right) \\ \frac{e^\theta}{1-e^\theta} & = \frac{\mu}{\nu}\end{cases}$$

$$b(\theta) = \nu\log\left(1+\frac{e^\theta}{1-e^\theta}\right) = \nu\log\left(\frac{1}{1-e^\theta}\right) = -\nu\log\left(1-e^\theta\right)$$

$$\mathbb{E}(Y) = \frac{db(\theta)}{d(\theta)} = \frac{\nu e^\theta}{1-e^\theta} = \nu\frac{\mu}{\nu} = \mu \qquad (1)$$

$$\mathbb{E}(Z) = \frac{(1-\pi)}{1-f_y(0;\mu,\nu)}\mathbb{E}(Y) = \frac{(1-\pi)}{1-f_y(0;\mu,\nu)}\mu = \frac{(1-\pi)\mu}{1-\left(\frac{\nu}{\mu+\nu}\right)^\nu}$$

$\Longrightarrow$ Variância de Z $: Var(Z) = \mathbb{E}(Z^2) - \mathbb{E}^2(Z)$

$$\mathbb{E}(Z^2) = \sum_{z_i=0}^{+\infty} z^2 f_z(z_i;\mu,\nu,\pi) = \ldots = \frac{(1-\pi)}{1-f_y(0;\mu,\nu)}\mathbb{E}(Y^2) = \frac{(1-\pi)}{1-f_y(0;\mu,\nu)}(Var(Y)+\mathbb{E}^2(Y)) \text{ vide E(Z) } (**)$$

$$Var(Y) = \phi^{-1}\frac{d^2b(\theta)}{d(\theta^2)} = 1\cdot\nu\frac{e^\theta(1-e^\theta)+e^\theta\cdot e^\theta}{(1-e^\theta)^2} = \frac{\nu e^\theta}{(1-e^\theta)^2} = \frac{\nu\mu}{\nu(1-e^\theta)} = \frac{\mu}{(1-\frac{\mu}{\mu+\nu})} = \frac{\mu(\mu+\nu)}{\nu} \qquad (2)$$

$$\mathbb{E}(Z^2) = \frac{(1-\pi)}{1-f_y(0;\mu,\nu)}(Var(Y)+\mathbb{E}^2(Y))$$

$$\mathbb{E}(Z^2) = \frac{(1-\pi)}{1-\left(\frac{\nu}{\mu+\nu}\right)^\nu}\left[\frac{\mu(\mu+\nu)}{\nu}+\mu^2\right], \qquad \mathbb{E}(Y^2) = \frac{\mu(\mu+\nu)}{\nu}+\mu^2 \qquad (3)$$

$$Var(Z) = \mathbb{E}(Z^2) - \mathbb{E}^2(Z)$$

$$Var(Z) = \frac{(1-\pi)}{1-\left(\frac{\nu}{\mu+\nu}\right)^\nu}\left[\frac{\mu(\mu+\nu)}{\nu}+\mu^2\right] - \left[\frac{(1-\pi)\mu}{1-\left(\frac{\nu}{\mu+\nu}\right)^\nu}\right]^2$$

$$Var(Z) = \frac{(1-\pi)}{1-\left(\frac{\nu}{\mu+\nu}\right)^\nu}\left[\frac{\mu(\mu+\nu)}{\nu}+\mu^2 - \frac{(1-\pi)\mu^2}{1-\left(\frac{\nu}{\mu+\nu}\right)^\nu}\right]$$

$$Var(Z) = \frac{(1-\pi)}{1-\left(\frac{\nu}{\mu+\nu}\right)^\nu}\left[\frac{\mu^2}{\nu}+\mu+\mu^2 - \frac{(1-\pi)\mu^2}{1-\left(\frac{\nu}{\mu+\nu}\right)^\nu}\right]$$

$$Var(Z) = \frac{(1-\pi)\mu}{1-\left(\frac{\nu}{\mu+\nu}\right)^\nu} + \frac{(1-\pi)\mu^2}{1-\left(\frac{\nu}{\mu+\nu}\right)^\nu}\left[1+\frac{1}{\nu} - \frac{(1-\pi)}{1-\left(\frac{\nu}{\mu+\nu}\right)^\nu}\right]$$

$\Longrightarrow$ Se $U \sim$ ZINBI (μ, ν, π)

$$f_u(u_i; \mu, \nu, \pi) = \begin{cases} \pi + (1-\pi)P(y=0) & \text{se } u_i = 0 \\ (1-\pi)f_y(u_i; \mu, \nu) & \text{se } u_i = 1, 2, \ldots, \end{cases}$$

em que $f_y(u_i; \mu, \nu)$ denota a função de probabilidade de uma BN (μ, ν).

$\Longrightarrow$ Condição 1) $f_u(u_i; \mu, \nu, \pi) \geq 0$

$$f_y(u; \mu, \nu) = \frac{\Gamma(u+\nu)}{\Gamma(\nu)\Gamma(u+1)} \left(\frac{\mu}{\mu+\nu}\right)^u \left(\frac{\nu}{\mu+\nu}\right)^\nu$$

$$P(y=0) = f_y(0; \mu, \nu) = \left(\frac{\nu}{\mu+\nu}\right)^\nu$$

$$\sum_{i=1}^{+\infty} f_y(u_i; \mu, \nu) = 1 - f_y(0; \mu, \nu) \text{ e } 0 \leq f_y(0; \mu, \nu) \leq 1$$

Se $f_y(0; \mu, \nu) = 0,$ observa-se que $f_y(u_i; \mu, \nu) = 1$ e $f_u(u_i; \mu, \nu, \pi) = 1$

Se $f_y(0; \mu, \nu) = 1,$ observa-se que $f_y(u_i; \mu, \nu) = 0$ e $f_u(u_i; \mu, \nu, \pi) = 1$

Se $0 < f_y(0; \mu, \nu) < 1,$ observa-se que $0 < f_y(u_i; \mu, \nu) < 1$ e $0 \leq f_u(u_i; \mu, \nu, \pi) \leq 1$

Logo, $f_u(u_i; \mu, \nu, \pi) \geq 0$

$\Longrightarrow$ Condição 2) : $\sum_{i=0}^{+\infty} f_y(u_i; \mu, \nu, \pi) = 1$

$$\sum_{i=0}^{+\infty} f_y(u_i; \mu, \nu, \pi) = f_y(0; \mu, \nu, \pi) + \sum_{i=1}^{+\infty} f_y(u_i; \mu, \nu, \pi) = \pi + (1-\pi)P(y=0) + \sum_{i=1}^{+\infty} f_y(u_i; \mu, \nu, \pi)$$

$$\sum_{i=0}^{+\infty} f_y(u_i; \mu, \nu, \pi) = \pi + (1-\pi)P(y=0) + (1-\pi)\sum_{i=1}^{+\infty} f_y(u_i; \mu, \nu)$$

$$\sum_{i=0}^{+\infty} f_y(u_i; \mu, \nu, \pi) = \pi + (1-\pi)P(y=0) + (1-\pi)\left[1 - f_y(0; \mu, \nu)\right] = \pi + (1-\pi)\left[P(y=0) + 1 - P(y=0)\right]$$

$$\sum_{i=0}^{+\infty} f_y(u_i; \mu, \nu, \pi) = \pi + 1 - \pi = 1, \qquad \text{lembrando que: } f_y(0; \mu, \nu) = P(y=0)$$

Como satisfez as duas condições, $f_u(u_i; \mu, \nu, \pi)$ é uma fp.

$\Longrightarrow$ Esperança de U

$$\mathbb{E}(U) = \sum_{u_i=0}^{+\infty} u f_u(u_i; \mu, \nu, \pi) = \sum_{u_i=0}^{+\infty} u f_u(u_i; \mu, \nu, \pi) = 0 \cdot f_u(0; \mu, \nu, \pi) + \sum_{u_i=1}^{+\infty} u f_u(u_i; \mu, \nu, \pi)$$

$$\mathbb{E}(U) = (1-\pi)\sum_{u_i=1}^{+\infty} u f_y(u_i; \mu, \nu) = (1-\pi)\left[E(Y) - 0 \cdot f_y(0; \mu, \nu)\right] = (1-\pi)E(Y)$$

vide (1) e lembre-se que , $E(Y) = \sum_{y=0}^{+\infty} y f_y(0; \mu, \nu) = 0 \cdot f_y(u_i; \mu, \nu) + \sum_{y=1}^{+\infty} y f_y(u_i; \mu, \nu)$

$$\mathbb{E}(U) = (1-\pi)\mu$$

$\Longrightarrow$ Variância de U: $Var(U) = \mathbb{E}(U^2) - \mathbb{E}^2(U)$

$$\mathbb{E}(U^2) = \sum_{u_i=0}^{+\infty} u^2 f_u(u_i;\mu,\nu,\pi) = 0 \cdot f_u(0;\mu,\nu,\pi) + \sum_{u_i=1}^{+\infty} u^2 f_u(u_i;\mu,\nu,\pi) = \sum_{u_i=1}^{+\infty} u^2 (1-\pi) f_y(u_i;\mu,\nu)$$

$$\mathbb{E}(U^2) = (1-\pi)\sum_{u_i=1}^{+\infty} u^2 f_y(u_i;\mu,\nu) = (1-\pi)\left[\mathbb{E}(Y^2) - 0\cdot f_y(0;\mu,\nu)\right] = (1-\pi)\underbrace{\mathbb{E}(Y^2)}_{\text{vide (3)}} = (1-\pi)\left[\frac{\mu(\mu+\nu)}{\nu} + \mu^2\right]$$

$$\begin{aligned}
Var(U) &= \mathbb{E}(U^2) - \mathbb{E}^2(U)\\
Var(U) &= (1-\pi)\left[\frac{\mu(\mu+\nu)}{\nu} + \mu^2\right] - [(1-\pi)\mu]^2\\
Var(U) &= (1-\pi)\left[\frac{\mu^2}{\nu} + \mu + \mu^2\right] - [(1-\pi)\mu]^2\\
Var(U) &= (1-\pi)\mu + (1-\pi)\mu^2\left[\frac{1}{\nu} + \pi\right]
\end{aligned}$$

5. A função de probabilidade da Poisson truncada em zero é expressa na forma

$$f(y;\lambda) = \frac{e^{-\lambda}\lambda^y}{y!(1-e^{-\lambda})}, \quad y = 1, 2, \ldots, \quad \lambda > 0$$

Obter $E(Y)$ e $\mathrm{Var}(Y)$. Para uma amostra de tamanho n, desenvolver um processo iterativo para obter a estimativa de máxima verossimilhança $\hat{\lambda}$. Obter também a variância assintótica $\mathrm{Var}(\hat{\lambda})$.

Resolução

$$f(y;\lambda) = \frac{e^{-\lambda}\lambda^y}{y!(1-e^{-\lambda})} = \frac{e^{-\lambda}e^{\log(\lambda^y)}}{e^{\log\left[y!(1-e^{-\lambda})\right]}} = \exp\left\{-\lambda + \log(\lambda^y) - \log[y!(1-e^{-\lambda})]\right\}$$

$$f(y;\lambda) = \exp\left\{y\log(\lambda) - \left[\lambda + \log(1-e^{-\lambda})\right] + [-\log(y!)]\right\}$$

$$f(y;\lambda) \text{ pertence à família exponencial de } \begin{cases} \phi &= 1\\ \theta &= \log(\lambda)\\ y_i &= y\\ b(\theta) &= \lambda + \log(1-e^{-\lambda})\\ c(y,\phi) &= -\log(y!)] \end{cases}$$

$$\lambda = e^\theta, \quad b(\theta) = e^\theta + \log(1-e^{-e^\theta})$$

$$\mathbb{E}(Y) = \mu = \frac{db(\theta)}{d\theta} = e^\theta + \frac{e^{-e^\theta}\cdot e^\theta}{1-e^{-e^\theta}} = \lambda + \frac{e^{-\lambda}\cdot\lambda}{1-e^{-\lambda}} = \frac{\lambda - \lambda e^{-\lambda} + \lambda e^{-\lambda}}{1-e^{-\lambda}} = \frac{\lambda}{1-e^{-\lambda}} = \frac{\lambda e^\lambda}{e^\lambda - 1}$$

$$Var(Y) = \phi^{-1}V(\mu) = (1)^{-1}\frac{d^2b(\theta)}{d\theta^2} = \frac{d}{d\theta}\left(\frac{e^\theta}{1-e^{-e^\theta}}\right) = \frac{e^\theta(1-e^{-e^\theta}) - (-e^{-e^\theta}(-e^\theta)e^\theta)}{\left(1-e^{-e^\theta}\right)^2}$$

$$Var(Y) = \frac{e^{\theta}(1-e^{-e^{\theta}}) - (-e^{-e^{\theta}}(-e^{\theta})e^{\theta})}{\left(1-e^{-e^{\theta}}\right)^2} = \frac{e^{\theta}(1-e^{-e^{\theta}}) - e^{-e^{\theta}}e^{\theta}e^{\theta}}{\left(1-e^{-e^{\theta}}\right)^2} = \frac{\lambda(1-e^{-\lambda}) - e^{-\lambda}\lambda\cdot\lambda}{(1-e^{-\lambda})^2}$$

$$Var(Y) = \frac{\lambda}{1-e^{-\lambda}}\left(\frac{1-e^{-\lambda}}{1-e^{-\lambda}} - \frac{e^{-\lambda}\lambda}{1-e^{-\lambda}}\right) = \mu(1-\mu e^{-\lambda})$$

$$\mu = \frac{\lambda}{1-e^{-\lambda}}$$

$$1-e^{-\lambda} = \frac{\lambda}{\mu}$$

$$1-\frac{\lambda}{\mu} = e^{-\lambda} \iff \mu - \lambda = \mu e^{-\lambda}$$

$$Var(Y) = \mu\left[1-(\mu-\lambda)\right] = \mu(1-\mu+\lambda), \text{ podendo ser escrita também como:}$$

$$Var(Y) = \frac{\lambda}{1-e^{-\lambda}}\left(1 - \frac{e^{-\lambda}\lambda}{1-e^{-\lambda}}\right)$$

$\Longrightarrow$Estimativa de máxima verossimilhança

$$f(y;\lambda) = \frac{e^{-\lambda}\lambda^{y}}{y!(1-e^{-\lambda})}$$

$$f(\lambda;y) \stackrel{iid}{=} \prod_{i=1}^{n}\frac{e^{-\lambda}\lambda^{y_i}}{y_i!(1-e^{-\lambda})} = e^{-n\lambda}\cdot\left(\frac{1}{1-e^{-\lambda}}\right)^{n}\lambda^{\sum_{i=1}^{n}y_i}\prod_{i=1}^{n}\frac{1}{y_i!}$$

$$\ell(\lambda) = -n\lambda - n\log(1-e^{-\lambda}) + \log(\lambda)\sum_{i=1}^{n}y_i - \sum_{i=1}^{n}\log(y_i!)$$

$$\frac{d\ell(\lambda)}{d\lambda} = -n - \frac{ne^{-\lambda}}{1-e^{-\lambda}} + \frac{1}{\lambda}\sum_{i=1}^{n}y_i$$

$$\frac{d\ell(\lambda)}{d\lambda} = 0$$

$$-n - \frac{ne^{-\hat{\lambda}}}{1-e^{-\hat{\lambda}}} + \frac{1}{\hat{\lambda}}\sum_{i=1}^{n}y_i = 0$$

$$-n - \frac{ne^{-\hat{\lambda}}}{1-e^{-\hat{\lambda}}} + \frac{n}{\hat{\lambda}}\bar{y} = 0$$

$$-n - \frac{n}{e^{\hat{\lambda}}-1} + \frac{n}{\hat{\lambda}}\bar{y} = 0$$

$$g(\lambda) = -n - \frac{n}{e^{\lambda}-1} + \frac{n}{\lambda}\bar{y}$$

$$G(\lambda) = \frac{dg(\lambda)}{d\lambda} = \frac{n\,e^{\lambda}}{(e^{\lambda}-1)^2} - \frac{n}{\lambda^2}\bar{y}$$

$$\lambda^{(m+1)} = \lambda^{(m)} - \frac{g[\lambda^{(m)}]}{G[\lambda^{(m)}]}, \quad m = 0,1,2,\ldots$$

$$\lambda^{(m+1)} = \lambda^{(m)} - \frac{-n - \frac{n}{e^{\lambda^{(m)}}-1} + \frac{n\bar{y}}{\lambda^{(m)}}}{\frac{ne^{\lambda^{(m)}}}{[e^{\lambda^{(m)}}-1]^2} - \frac{n\bar{y}}{[\lambda^{(m)}]^2}}$$

O sobrescrito (m) indica que as expressões são avaliadas no valor de λ na iteração m. Repetimos o procedimento até que o seguinte critério de convergência seja satisfeito:

$$\left|\frac{\lambda^{(m+1)} - \lambda^{(m)}}{\lambda^{(m)}}\right| < \epsilon,$$

para algum $\epsilon > 0$ suficientemente pequeno.

$\Longrightarrow$ Estimador de máxima verossimilhança ($\hat{\lambda}$) de λ

Seja, $\bar{y}$, a média amostral da Poisson não truncada, o chute inicial ($\lambda^{(0)} = \bar{y}$). O estimador de máxima verossimilhança $\hat{\lambda}$ será aquele encontrado em ou após $m+1$ iterações, nas quais se alcança a convergência.

$\Longrightarrow$ Variância assintótica

$$
\begin{aligned}
Var(\hat{\lambda}) &= \frac{1}{K_{\lambda\lambda}}\\
K_{\lambda\lambda} &= -\mathbb{E}\left\{\frac{d^2L(\lambda)}{d\lambda^2}\right\} = Var\left(U_\lambda = \frac{dL(\lambda)}{d\lambda}\right)\\
U_\lambda &= \frac{dL(\lambda)}{d\lambda} - n - \frac{ne^{-\lambda}}{1-e^{-\lambda}} + \frac{1}{\lambda}\sum_{i=1}^{n} y_i\\
K_{\lambda\lambda} &= Var(U_\lambda) = Var\left(-n - \frac{ne^{-\lambda}}{1-e^{-\lambda}} + \frac{1}{\lambda}\sum_{i=1}^{n} y_i\right)\\
K_{\lambda\lambda} &= \frac{1}{\lambda^2}\sum_{i=1}^{n} Var(y) = \frac{1}{\lambda^2}\sum_{i=1}^{n}\frac{\lambda}{1-e^{-\lambda}}\left(1-\frac{e^{-\lambda}\lambda}{1-e^{-\lambda}}\right)\\
K_{\lambda\lambda} &= \frac{1}{\lambda^2}\frac{n\lambda e^{\lambda}}{e^{\lambda}-1}\left(1-\frac{\lambda}{e^{\lambda}-1}\right)\\
K_{\lambda\lambda} &= \frac{ne^{\lambda}}{\lambda(e^{\lambda}-1)} - \frac{ne^{\lambda}}{(e^{\lambda}-1)^2}\\
K_{\lambda\lambda} &= \frac{ne^{\lambda}(e^{\lambda}-1) - n\lambda e^{\lambda}}{\lambda(e^{\lambda}-1)^2}\\
K_{\lambda\lambda} &= \frac{n(e^{2\lambda}-e^{\lambda}-\lambda e^{\lambda})}{\lambda(e^{\lambda}-1)^2}\\
Var(\hat{\lambda}) &= \frac{1}{K_{\lambda\lambda}}\\
Var(\hat{\lambda}) &= \frac{1}{\frac{n(e^{2\hat{\lambda}}-e^{\hat{\lambda}}-\hat{\lambda}e^{\hat{\lambda}})}{\hat{\lambda}(e^{\hat{\lambda}}-1)^2}}\\
Var(\hat{\lambda}) &= \frac{\hat{\lambda}(e^{\hat{\lambda}}-1)^2}{n(e^{2\hat{\lambda}}-e^{\hat{\lambda}}-\hat{\lambda}e^{\hat{\lambda}})},
\end{aligned}
$$

em que $\hat{\lambda}$ é o estimador de máxima verossimilhança encontrado em ou após $m+1$ iterações, nas quais se alcança a convergência.

6. Supor $Y_{ij} \stackrel{iid}{\sim} Ge(\mu_i)$ (distribuição geométrica), em que $\mu_i = \mathbb{E}(Y_i)$ e $Var(Y_i) = V(\mu_i) = \mu_i(\mu_i - 1), i = 1, 2,$ e $j = 1, \ldots, m$, com $\log(\mu_1) = \eta_1 = \alpha$ e $\log(\mu_2) = \eta_2 = \alpha + \Delta$. Como ficam as matrizes $\mathbf{X}$ e $\mathbf{W}$? Obter as variâncias assintóticas $Var(\hat{\alpha})$ e $Var(\hat{\Delta})$ além da covariância assintótica $Cov(\hat{\alpha}, \hat{\Delta})$, deixando as expressões em função dos pesos $\mathbf{W}$. Mostre que a estatística do teste de escore para testar $H_0 : \Delta = 0$ contra $H_1 : \Delta \neq 0$ fica expressa na forma
$$\xi_{SR} = \frac{m(\bar{y}_2 - \bar{y}_1)^2}{2\bar{y}(\bar{y} - 1)}.$$
Qual a distribuição nula assintótica da estatística do teste?

Resolução

(i) Matrizes X e W:
Uma vez que $\eta = X\beta$, e

$$\eta = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_1 \\ \eta_2 \\ \vdots \\ \eta_2 \end{bmatrix}_{2m\times 1} = \begin{bmatrix} \alpha \\ \vdots \\ \alpha \\ \alpha+\Delta \\ \vdots \\ \alpha+\Delta \end{bmatrix}_{2m\times 1} \quad \text{e} \quad \beta = \begin{bmatrix} \alpha \\ \Delta \end{bmatrix}_{2\times 1},$$

segue que

$$X = \begin{bmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{bmatrix}_{2m\times 2}.$$

Como $\omega_i = \frac{(d\mu_i/d\eta_i)^2}{V(\mu_i)}$,

Para $i = 1$:
$d\mu_1/d\eta_1 = e^{\eta_1} = e^{\alpha}$, $V(\mu_1) = e^{\alpha}(e^{\alpha} - 1)$ e $\omega_1 = \frac{e^{\alpha}}{(e^{\alpha}-1)}$.

Para $i = 2$:
$d\mu_2/d\eta_2 = e^{\eta_2} = e^{\alpha+\Delta}$, $V(\mu_2) = e^{\alpha+\Delta}(e^{\alpha+\Delta} - 1)$ e $\omega_2 = \frac{e^{\alpha+\Delta}}{(e^{\alpha+\Delta}-1)}$.

Logo, segue que
$W = \text{diag}\{\omega_1, \cdots, \omega_1, \omega_2, \cdots, \omega_2\}$, e $V = \text{diag}\{V(\mu_1), \cdots, V(\mu_1), V(\mu_2), \cdots, V(\mu_2)\}$.

(ii) Variâncias Assintóticas:
Vale que a variância assintótica do vetor $\hat{\beta}$ é dada por

$$Var(\hat{\beta}) = \begin{bmatrix} Var(\hat{\beta}) & Cov(\hat{\beta}, \hat{\Delta}) \\ Cov(\hat{\beta}, \hat{\Delta}) & Var(\hat{\Delta}) \end{bmatrix} = \phi^{-1}(X^\top W X)^{-1}$$

Segue que

$$\phi = 1 \Rightarrow \phi^{-1} = 1$$

$$X^\top W = \begin{bmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ 0 & \cdots & 0 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \omega_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega_2 \end{bmatrix} = \begin{bmatrix} \omega_1 & \cdots & \omega_1 & \omega_2 & \cdots & \omega_2 \\ 0 & \cdots & 0 & \omega_2 & \cdots & \omega_2 \end{bmatrix}$$

$$X^\top WX = \begin{bmatrix} \omega_1 & \cdots & \omega_1 & \omega_2 & \cdots & \omega_2 \\ 0 & \cdots & 0 & \omega_2 & \cdots & \omega_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} m(\omega_1+\omega_2) & m\omega_2 \\ m\omega_2 & m\omega_2 \end{bmatrix}$$

$$(X^\top WX)^{-1} = \frac{1}{m^2(\omega_1+\omega_2)\omega_2 - m^2\omega_2^2} \begin{bmatrix} m\omega_2 & -m\omega_2 \\ -m\omega_2 & m(\omega_1+\omega_2) \end{bmatrix}$$

$$(X^\top WX)^{-1} = \frac{1}{m^2\omega_1\omega_2} \begin{bmatrix} m\omega_2 & -m\omega_2 \\ -m\omega_2 & m(\omega_1+\omega_2) \end{bmatrix}.$$

Logo, segue que

$Var(\hat{\beta}) = \frac{1}{m\omega_1}$, $Var(\hat{\Delta}) = \frac{\omega_1+\omega_2}{m\omega_1\omega_2}$, e $Cov(\hat{\beta},\hat{\Delta}) = -\frac{1}{m\omega_1}$.

(iii) Estatística do teste de escore:
Sob as hipóteses $H_0 : \Delta = 0$ contra $H_1 : \Delta \neq 0$, a estatística do teste de escore é

$$\xi_{SR} = \left[U_\Delta^0\right]^2 \hat{Var}_0(\hat{\Delta}).$$

Segue que

$$\Delta = 0 \Rightarrow \omega_2 = \omega_1 \Rightarrow Var_0(\hat{\Delta}) = 2\omega_1/m\omega_1^2 = 2/m\omega_1$$

Ainda

$$U_\Delta = \phi X_2^\top W^{1/2}V^{-1/2}(y-\mu) = X_2^\top W^{1/2}V^{-1/2}(y-\mu)$$

$$X_2^\top = \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \end{bmatrix}$$

$$X_2^\top W^{1/2} = \begin{bmatrix} 0 & \cdots & 0 & \sqrt{\omega_2} & \cdots & \sqrt{\omega_2} \end{bmatrix}$$

$$X_2^\top W^{1/2}V^{-1/2} = \begin{bmatrix} 0 & \cdots & 0 & \sqrt{\omega_2/V(\mu_2)} & \cdots & \sqrt{\omega_2/V(\mu_2)} \end{bmatrix}$$

$$X_2^\top W^{1/2}V^{-1/2}(y-\mu) = \sum_{i=m+1}^{2m} y_i\sqrt{\omega_2/V(\mu_2)} - \sum_{i=m+1}^{2m} \mu_2\sqrt{\omega_2/V(\mu_2)}$$

$$X_2^\top W^{1/2}V^{-1/2}(y-\mu) = m\bar{y}_2\sqrt{\omega_2/V(\mu_2)} - m\mu_2\sqrt{\omega_2/V(\mu_2)}.$$

$$X_2^\top W^{1/2}V^{-1/2}(y-\mu) = \frac{m\bar{y}_2}{\mu_2-1} - \frac{m\mu_2}{\mu_2-1}.$$

Sob H_0,

$$\mu_1 = \mu_2 \Rightarrow \hat{\mu}_0 = \bar{y}.$$

Portanto,

$$\xi_{SR} = \left[\frac{m}{\bar{y}-1}(\bar{y}_2 - \bar{y})\right]^2 \frac{2(\bar{y}-1)}{m\bar{y}}$$

$$\xi_{SR} = \frac{2m(\bar{y_2} - \bar{y})^2}{(\bar{y} - 1)\bar{y}}.$$

Como $\bar{y} = \frac{\bar{y_1} + \bar{y_2}}{2}$,

$$\xi_{SR} = \frac{2m(\bar{y_2}/2 - \bar{y_1}/2)^2}{(\bar{y} - 1)\bar{y}}$$

$$\xi_{SR} = \frac{m(\bar{y_2} - \bar{y_1})^2}{2\bar{y}(\bar{y} - 1)}$$

$$\xi_{SR} \underset{m \to \infty}{\overset{H_0}{\sim}} \chi^2_{(1)}$$

Sob a hipótese nula $H_0 : \Delta = 0$, quando o tamanho da amostra tende ao infinito ($m \to \infty$), a estatística do teste da razão de verossimilhanças segue, assintoticamente, uma distribuição qui-quadrado com 1 grau de liberdade ($\chi^2_{(1)}$), pois estamos testando um único parâmetro (Δ).