# UNIVERSIDADE DE SÃO PAULO
# Programa de Pos-Graduação em Estatística

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**Lista 02 – Semestre de 2024-II – Prof. Gilberto**

1. Supor $Y_{ij} \overset{\text{ind}}{\sim} \text{N}(\mu, \phi_i)$, para $i = 1, 2$ e $j = 1, \ldots, r$. Mostre que a estatística do teste da razão de verossimilhanças para testar $\text{H}_0 : \phi_1 = \phi_2$ contra $\text{H}_1 : \phi_1 \neq \phi_2$ pode ser expressa na forma

$$\xi_{RV} = r \log\left(\frac{\hat{\phi}_1\hat{\phi}_2}{\hat{\phi}^2}\right),$$

em que $\hat{\phi}$, $\hat{\phi}_1$ e $\hat{\phi}_2$ são as estimativas de máxima verossimilhança, com $\hat{\phi}_1 = \frac{r}{D_1}$, $\hat{\phi}_2 = \frac{r}{D_2}$ e $\hat{\phi} = \frac{2r}{D_0}$. Qual a distribuição nula assintótica da estatística do teste?

**Resolução**

$$\begin{aligned} f(y_{ij}, \phi_i) &= \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}(y_{ij} - \mu)^2\right\}, \quad \phi_i = \frac{1}{\sigma^2} \Longrightarrow \sqrt{\phi_i} = \frac{1}{\sigma}, \sigma > 0, \phi_i > 0 \\ &= \frac{\sqrt{\phi_i}}{\sqrt{2\pi}} \exp\left\{-\frac{\phi_i}{2}(y_{ij} - \mu)^2\right\}. \end{aligned}$$

Como as variáveis aleatórias são independentes e identicamente distribuídas, a função de verossimilhança é dada por

$$L(\mu, \phi_i) \overset{ind}{=} \prod_{i=1}^{2}\prod_{j=1}^{r}\left(\frac{\sqrt{\phi_i}}{\sqrt{2\pi}}\right) \exp\left\{-\sum_{i=1}^{2}\sum_{j=1}^{r}\frac{\phi_i}{2}(y_{ij} - \mu)^2\right\}.$$

Assim, o logaritmo da função de verossimilhança é dado por

$$\ell(\mu, \phi_i) = \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{r}\log(\phi_i) - \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{r}\log(2\pi) - \sum_{i=1}^{2}\sum_{j=1}^{r}\frac{\phi_i}{2}(y_{ij} - \mu)^2.$$

$$\frac{d\ell(\mu, \phi_i)}{d\mu} = -2\sum_{i=1}^{2}\sum_{j=1}^{r}\frac{\hat{\phi}_i}{2}(y_{ij} - \hat{\mu})(-1) \text{ e } \frac{d^2\ell(\mu, \phi_i)}{d\mu^2} = -\sum_{i=1}^{2}\sum_{j=1}^{r}\hat{\phi}_i y_{ij} < 0$$

$$\frac{d\ell(\mu, \phi_i)}{d\mu} = 0$$

$$-2\sum_{i=1}^{2}\sum_{j=1}^{r}\frac{\hat{\phi}_i}{2}(y_{ij} - \hat{\mu})(-1) = \sum_{i=1}^{2}\sum_{j=1}^{r}\hat{\phi}_i(y_{ij} - \hat{\mu}) = 0 \Longrightarrow \sum_{i=1}^{2}\sum_{j=1}^{r}\hat{\phi}_i\hat{\mu} = \sum_{i=1}^{2}\sum_{j=1}^{r}\hat{\phi}_i y_{ij}$$

$$\hat{\mu} = \frac{\sum_{i=1}^{2}\sum_{j=1}^{r}\hat{\phi}_i y_{ij}}{\sum_{i=1}^{2}\sum_{j=1}^{r}\hat{\phi}_i} = \frac{\hat{\phi}_1\sum_{j=1}^{r}y_{1j} + \hat{\phi}_2\sum_{j=1}^{r}y_{2j}}{\sum_{j=1}^{r}\hat{\phi}_1 + \sum_{j=1}^{r}\hat{\phi}_2} = \frac{\hat{\phi}_1 r\bar{y}_1 + \hat{\phi}_2 r\bar{y}_2}{r\hat{\phi}_1 + r\hat{\phi}_2} = \frac{\hat{\phi}_1\bar{y}_1 + \hat{\phi}_2\bar{y}_2}{\hat{\phi}_1 + \hat{\phi}_2}, \text{ e,}$$

$$\ell(\phi_1, \phi_2) = \frac{1}{2}\sum_{j=1}^{r}\log(\phi_1) + \frac{1}{2}\sum_{j=1}^{r}\log(\phi_2) - \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{r}\log(2\pi) - \sum_{j=1}^{r}\frac{\phi_1}{2}(y_{1j} - \mu)^2 - \sum_{j=1}^{r}\frac{\phi_2}{2}(y_{2j} - \mu)^2$$

$$\Longrightarrow \begin{cases} \frac{d\ell(\phi_1,\phi_2)}{d\phi_1} = \frac{r}{2\phi_1} - \sum_{j=1}^{r} \frac{1}{2}(y_{1j}-\mu)^2 \\ \frac{d\ell(\phi_1,\phi_2)}{d\phi_2} = \frac{r}{2\phi_2} - \sum_{j=1}^{r} \frac{1}{2}(y_{2j}-\mu)^2 \end{cases} \quad \text{e} \quad \begin{cases} \frac{d^2\ell(\phi_1,\phi_2)}{d\phi_1^2} = -\frac{r}{2\phi_1^2} < 0 \\ \frac{d^2\ell(\phi_1,\phi_2)}{d\phi_2^2} = -\frac{r}{2\phi_2^2} < 0 \end{cases}$$

$$\frac{d\ell(\phi_i)}{d\phi_i} = 0,\ i=1,2 \Longrightarrow \begin{cases} \frac{r}{2\hat{\phi}_1} = \sum_{j=1}^{r} \frac{1}{2}(y_{1j}-\hat{\mu})^2 \\ \frac{r}{2\hat{\phi}_2} = \sum_{j=1}^{r} \frac{1}{2}(y_{2j}-\hat{\mu})^2 \end{cases} \Longleftrightarrow \begin{cases} \hat{\phi}_1 = \frac{r}{\sum_{j=1}^{r}(y_{1j}-\hat{\mu})^2} \\ \hat{\phi}_2 = \frac{r}{\sum_{j=1}^{r}(y_{2j}-\hat{\mu})^2} \end{cases} \Longrightarrow \begin{cases} \hat{\phi}_1 = \frac{r}{D_1} \\ \hat{\phi}_2 = \frac{r}{D_2} \end{cases}$$

$$\text{Sob } H_0 : \phi_1 = \phi_2 \Longrightarrow \hat{\phi}_1 = \hat{\phi}_2 \text{ e } \hat{\mu}^0 = \frac{\hat{\phi}\bar{y}_1 + \hat{\phi}\bar{y}_2}{\hat{\phi}+\hat{\phi}} = \frac{\bar{y}_1+\bar{y}_2}{2} = \bar{y},\ \text{e },$$

$$\ell(\mu,\phi) = \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{r}\log(\phi) - \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{r}\log(2\pi) - \sum_{i=1}^{2}\sum_{j=1}^{r}\frac{\phi}{2}(y_{ij}-\mu)^2$$

$$\frac{d\ell(\mu,\phi)}{d\phi} = \sum_{i=1}^{2}\sum_{j=1}^{r}\frac{1}{2\phi} - \sum_{i=1}^{2}\sum_{j=1}^{r}\frac{1}{2}(y_{ij}-\mu)^2 = \frac{2r}{2\phi} - \sum_{i=1}^{2}\sum_{j=1}^{r}\frac{1}{2}(y_{ij}-\mu)^2 = \frac{r}{\phi} - \sum_{i=1}^{2}\sum_{j=1}^{r}\frac{1}{2}(y_{ij}-\mu)^2$$

$$\frac{d\ell(\mu,\phi_i)}{d\phi} = 0 \Longrightarrow \frac{r}{\hat{\phi}} = \sum_{i=1}^{2}\sum_{j=1}^{r}\frac{1}{2}(y_{ij}-\hat{\mu}^0)^2 \Longleftrightarrow \hat{\phi} = \frac{2r}{\sum_{i=1}^{2}\sum_{j=1}^{r}(y_{ij}-\hat{\mu}^0)^2} = \frac{2r}{D_0}$$

$$\begin{aligned}
\xi_{RV} &= 2\left\{\ell(\hat{\phi}) - \ell(\phi^0)\right\} \\
&= 2\left\{\frac{1}{2}\sum_{j=1}^{r}\log\left(\hat{\phi}_1\right) + \frac{1}{2}\sum_{j=1}^{r}\log\left(\hat{\phi}_2\right) - \frac{\hat{\phi}_1}{2}\sum_{j=1}^{r}(y_{1j}-\hat{\mu})^2 - \frac{\hat{\phi}_2}{2}\sum_{j=1}^{r}(y_{2j}-\hat{\mu})^2\right. \\
&\left. - \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{r}\log\left(\hat{\phi}\right) + \frac{\hat{\phi}}{2}\sum_{i=1}^{2}\sum_{j=1}^{r}(y_{ij}-\hat{\mu}^0)^2\right\} \\
&= 2\left\{\frac{r}{2}\log\left(\hat{\phi}_1\right) + \frac{r}{2}\log\left(\hat{\phi}_2\right) - \frac{\hat{\phi}_1}{2}\frac{r}{\hat{\phi}_1} - \frac{\hat{\phi}_2}{2}\frac{r}{\hat{\phi}_2} - r\log\left(\hat{\phi}\right) + \frac{\hat{\phi}}{2}\frac{2r}{\hat{\phi}}\right\} \\
&= 2\left\{\frac{r}{2}\log\left(\hat{\phi}_1\hat{\phi}_2\right) - r - r\log\left(\hat{\phi}\right) + r\right\} = r\log\left(\hat{\phi}_1\hat{\phi}_2\right) - 2r\log\left(\hat{\phi}\right) \\
&= r\left[\log\left(\hat{\phi}_1\hat{\phi}_2\right) - \log\left(\hat{\phi}^2\right)\right] \\
&= r\log\left(\frac{\hat{\phi}_1\hat{\phi}_2}{\hat{\phi}^2}\right)
\end{aligned}$$

$$\xi_{RV} \underset{r\to\infty}{\overset{H_0}{\sim}} \chi^2_{(1)}.$$

2. Supor $Y_{ij} \overset{\text{ind}}{\sim} \mathrm{N}(\mu, \phi_i)$, para $i = 1, 2$ e $j = 1, \ldots, r$ em que $\log(\phi_1) = \lambda_1 = \alpha + \Delta$ e $\log(\phi_2) = \lambda_2 = \alpha - \Delta$. Como ficam as matrizes $\mathbf{Z}$ e $\mathbf{P}$? Obter $\hat{\alpha}$ e $\hat{\Delta}$ e as respectivas variâncias assintóticas, além de $\mathrm{Cov}(\hat{\alpha}, \hat{\Delta})$. Como fica a estatística do teste de Wald para testar $\mathrm{H}_0 : \Delta = 0$ contra $\mathrm{H}_1 : \Delta \neq 0$. Qual a distribuição nula assintótica da estatística do teste?

**Resolução**

Uma vez que $\boldsymbol{\lambda} = \mathbf{Z}\boldsymbol{\gamma}$, e

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_2 \end{bmatrix}_{2r \times 1} = \begin{bmatrix} \alpha + \Delta \\ \vdots \\ \alpha + \Delta \\ \alpha - \Delta \\ \vdots \\ \alpha - \Delta \end{bmatrix}_{2r \times 1} \quad \text{e} \quad \boldsymbol{\gamma} = \begin{bmatrix} \alpha \\ \Delta \end{bmatrix}_{2 \times 1},$$

segue que

$$\mathbf{Z} = \begin{bmatrix} 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ 1 & -1 \\ \vdots & \vdots \\ 1 & -1 \end{bmatrix}_{2r \times 2}.$$

Também, segue que $\mathbf{P} = V_\gamma H_\gamma^{-2}$, em que $V_\gamma = \mathrm{diag}\{-d''(\phi_1), \ldots, -d''(\phi_1), -d''(\phi_2), \ldots, -d''(\phi_2)\}_{2r \times 2r}$ e $H_\gamma = \mathrm{diag}\{h'(\phi_1), \ldots, h'(\phi_1), h'(\phi_2), \ldots, h'(\phi_2)\}_{2r \times 2r}$, sendo $h(\phi_i) = \lambda_i$. Uma vez que $Y_{ij} \overset{\text{ind}}{\sim} \mathrm{N}(\mu, \phi_i)$, segue que $d(\phi_i) = \frac{1}{2}\log(\phi_i) \implies d'(\phi_i) = \frac{1}{2\phi} \implies d''(\phi_i) = -\frac{1}{2\phi^2}, i = 1, 2.$ Também, $h(\phi_i) = \log(\phi_i) \implies h'(\phi_i) = \phi_i^{-1}, i = 1, 2$. Logo,

$$V_\gamma = \mathrm{diag}\{(2\phi_1^2)^{-1}, \ldots, (2\phi_1^2)^{-1}, (2\phi_2^2)^{-1}, \ldots, (2\phi_2^2)^{-1}\}_{2r \times 2r}, \quad \text{e}$$

$$H_\gamma = \mathrm{diag}\{\phi_1^{-1}, \ldots, \phi_1^{-1}, \phi_2^{-1}, \ldots, \phi_2^{-1}\} \implies H_\gamma^{-2} = \mathrm{diag}\{\phi_1^2, \ldots, \phi_1^2, \phi_2^2, \ldots, \phi_2^2\}_{2r \times 2r}, \quad \text{e portanto,}$$

$$\mathbf{P} = V_\gamma H_\gamma^{-2} = \mathrm{diag}\{1/2, \ldots, 1/2\}_{2r \times 2r} = \frac{1}{2}\mathrm{I}_{2r},$$

em que $\mathrm{I}_{2r}$ é a matriz identidade de ordem $2r$.

Pelo exercício 1), tem-se que os estimadores de máxima verossimilhança de $\phi_1$ e $\phi_2$ são dados por

$$\hat{\phi}_1 = r/D_1, \quad \text{e} \quad \hat{\phi}_2 = r/D_2,$$

em que $D_1 = \sum_{j=1}^r (y_{1j} - \hat{\mu})^2$ e $D_2 = \sum_{j=1}^r (y_{2j} - \hat{\mu})^2$. Pela invariância dos estimadores de máxima verossimilhança, segue que os estimadores de máxima verossimilhança de $\alpha$ e $\Delta$ são dados por

$$\exp\{\hat{\alpha} + \hat{\Delta}\} = r/D_1, \quad \exp\{\hat{\alpha} - \hat{\Delta}\} = r/D_2$$

$$\implies \hat{\alpha} + \hat{\Delta} = \log(r) - \log(D_1), \quad \hat{\alpha} - \hat{\Delta} = \log(r) - \log(D_2)$$

$$\implies \hat{\alpha} = \log(r) - \frac{log(D_1 D_2)}{2}, \quad \hat{\Delta} = \frac{\log(D_2/D_1)}{2}.$$

A matriz de informação de Fisher para o parâmetro $\gamma$ é dada por

$$K_{\gamma\gamma} = \mathbf{Z}^\top \mathbf{P} \mathbf{Z} = \frac{1}{2}\mathbf{Z}^\top \mathbf{Z} = \frac{1}{2}\begin{bmatrix} 2r & 0 \\ 0 & 2r \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix},$$

e sua inversa é dada por

$$K_{\gamma\gamma}^{-1} = \begin{bmatrix} r^{-1} & 0 \\ 0 & r^{-1} \end{bmatrix}.$$

Logo, segue que $\text{Var}(\hat{\alpha}) = r^{-1}$, $\text{Var}(\hat{\Delta}) = r^{-1}$ e $\text{Cov}(\hat{\alpha}, \hat{\Delta}) = 0$.

Para testar $\text{H}_0 : \Delta = \Delta_0 = 0$ contra $\text{H}_1 : \Delta \neq 0$, a estatística do teste de Wald é dada por

$$\xi_W = \frac{(\hat{\Delta} - \Delta_0)^2}{\text{Var}(\hat{\Delta})} = r\left[\frac{\log(D_2/D_1)}{2}\right]^2 = \frac{r}{4}[\log(D_2/D_1)]^2.$$

Uma vez que $\hat{\phi}_1 = r/D_1,\ \hat{\phi}_2 = r/D_2 \implies \hat{\phi}_1/\hat{\phi}_2 = D_2/D_1$, segue que

$$\xi_W = \frac{r}{4}[\log(\hat{\phi}_1/\hat{\phi}_2)]^2$$

$$\xi_W \underset{r\to\infty}{\overset{H_0}{\sim}} \chi^2_{(1)}.$$

3. Supor $Y_{ij} \overset{\text{ind}}{\sim} \mathrm{G}(\mu, \phi_i)$ com parte sistemática dada por $\log(\phi_i) = \lambda_i = \gamma z_i$, para $i = 1, \ldots, n$. Como ficam as matrizes $\mathbf{Z}$ e $\mathbf{P}$? Obter $\mathrm{U}_\gamma$ e $\mathrm{K}_{\gamma\gamma}$. Como fica a estatística do teste de escore para testar $\mathrm{H}_0 : \gamma = 0$ contra $\mathrm{H}_1 : \gamma \neq 0$. Qual a distribuição nula assintótica da estatística do teste?

**Resolução**

$$\boldsymbol{Z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}_{n\times 1} \quad \text{e } \boldsymbol{P} = \operatorname{diag}\{p_1, \ldots, p_n\},\ p_i = -d''(\phi_i)/h'(\phi_i)^2,\ i = 1, \ldots, n,\ \log(\phi_i) = \gamma z_i \Longrightarrow \phi_i = e^{\gamma z_i}$$

$$Y_{ij} \overset{\text{ind}}{\sim} \mathrm{G}(\mu, \phi_i),\ \text{ logo, } d''(\phi_i) = \phi_i^{-1} - \psi'(\phi_i)$$

$$h(\phi_i) = \lambda_i = \boldsymbol{z_i}\gamma = \log(\phi_i) \Longrightarrow h'(\phi_i) = \frac{1}{\phi_i} \text{ e } p_i = -\frac{d''(\phi_i)}{h'(\phi_i)^2} = -\frac{\left(\phi_i^{-1} - \psi'(\phi_i)\right)}{\left(\frac{1}{\phi_i}\right)^2} = \phi_i^2\psi'(\phi_i) - \phi_i$$

$$\boldsymbol{P} = \begin{bmatrix} \phi_1^2\psi'(\phi_1) - \phi_1 & 0 & \ldots & 0 \\ 0 & \phi_2^2\psi'(\phi_2) - \phi_2 & \ldots & 0 \\ \vdots & 0 & \phi_2^2\psi'(\phi_3) - \phi_3 & \vdots \\ 0 & \ldots & \ldots & \phi_n^2\psi'(\phi_n) - \phi_n \end{bmatrix}_{n\times n}$$

$$\boldsymbol{H}_\gamma = \operatorname{diag}\{h'(\phi_1), \ldots, h'(\phi_n)\} = \begin{bmatrix} \frac{1}{\phi_1} & 0 & \ldots & 0 \\ 0 & \frac{1}{\phi_2} & \ldots & 0 \\ \vdots & 0 & \frac{1}{\phi_3} & \vdots \\ 0 & \ldots & \ldots & \frac{1}{\phi_n} \end{bmatrix}_{n\times n} \quad \text{e } \boldsymbol{H}_\gamma^{-1} = \begin{bmatrix} \phi_1 & 0 & \ldots & 0 \\ 0 & \phi_2 & \ldots & 0 \\ \vdots & 0 & \phi_3 & \vdots \\ 0 & \ldots & \ldots & \phi_n \end{bmatrix}_{n\times n}$$

$$\boldsymbol{t} = (t_1, \ldots, t_n)^{\mathrm{T}} = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix}_{n\times 1}$$

$$\mu_T = (-d'(\phi_1), \ldots, -d'(\phi_n))^{\mathrm{T}} = \begin{bmatrix} -1 - \log(\phi_1) + \psi(\phi_1) \\ -1 - \log(\phi_2) + \psi(\phi_2) \\ \vdots \\ -1 - \log(\phi_n) + \psi(\phi_n) \end{bmatrix}_{n\times 1}, \ d'(\phi_i) = 1 + \log(\phi_1) - \psi(\phi_1)$$

$$\begin{aligned}
\mathrm{U}_\gamma &= \boldsymbol{Z}^{\mathrm{T}}\boldsymbol{H}_\gamma^{-1}(\boldsymbol{t} - \mu_T) \\
&= \begin{bmatrix} z_1 & z_2 & \ldots & z_n \end{bmatrix}_{1\times n} \begin{bmatrix} \phi_1 & 0 & \ldots & 0 \\ 0 & \phi_2 & \ldots & 0 \\ \vdots & 0 & \phi_3 & \vdots \\ 0 & \ldots & \ldots & \phi_n \end{bmatrix}_{n\times n} \left( \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix}_{n\times 1} - \begin{bmatrix} -1 - \log(\phi_1) + \psi(\phi_1) \\ -1 - \log(\phi_2) + \psi(\phi_2) \\ \vdots \\ -1 - \log(\phi_n) + \psi(\phi_n) \end{bmatrix}_{n\times 1} \right) \\
&= \begin{bmatrix} z_1\phi_1 & z_2\phi_2 & \ldots & \phi_n z_n \end{bmatrix}_{1\times n} \begin{bmatrix} t_1 + 1 + \log(\phi_1) - \psi(\phi_1) \\ t_2 + 1 + \log(\phi_2) - \psi(\phi_2) \\ \vdots \\ t_n + 1 + \log(\phi_n) - \psi(\phi_n) \end{bmatrix}_{n\times 1} \\
&= \sum_{i=1}^{n} z_i\phi_i \left[t_i + 1 + \log(\phi_i) - \psi(\phi_i)\right] \\
&= \sum_{i=1}^{n} z_i e^{\gamma z_i} \left[t_i + 1 + \gamma z_i - \psi(e^{\gamma z_i})\right]
\end{aligned}$$

$$
\begin{aligned}
K_{\gamma\gamma} &= \boldsymbol{Z}^T \boldsymbol{P} \boldsymbol{Z} \\
&= \begin{bmatrix} z_1 & z_2 & \dots & z_n \end{bmatrix}_{1\times n} \begin{bmatrix} \phi_1^2\psi'(\phi_1)-\phi_1 & 0 & \dots & 0 \\ 0 & \phi_2^2\psi'(\phi_2)-\phi_2 & \dots & 0 \\ \vdots & 0 & \phi_2^2\psi'(\phi_3)-\phi_3 & \vdots \\ 0 & \dots & \dots & \phi_n^2\psi'(\phi_n)-\phi_n \end{bmatrix}_{n\times n} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}_{n\times 1} \\
&= \begin{bmatrix} z_1(\phi_1^2\psi'(\phi_1)-\phi_1) & z_2(\phi_2^2\psi'(\phi_2)-\phi_2) & \dots & z_n(\phi_n^2\psi'(\phi_n)-\phi_n) \end{bmatrix}_{1\times n} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}_{n\times 1} \\
&= \sum_{i=1}^{n} z_i^2 [\phi_i^2\psi'(\phi_i)-\phi_i] \\
&= \sum_{i=1}^{n} z_i^2 [e^{2\gamma z_i}\psi'(e^{\gamma z_i})-e^{\gamma z_i}] \\
&= \sum_{i=1}^{n} z_i^2 e^{\gamma z_i}[e^{\gamma z_i}\psi'(e^{\gamma z_i})-1]
\end{aligned}
$$

$$
\begin{aligned}
\xi_{SR} =& \mathrm{U}_\gamma\left(\hat{\gamma}^0\right)^T \hat{\mathrm{Var}}_0\left(\hat{\gamma}\right) \mathrm{U}_\gamma\left(\hat{\gamma}^0\right) = \mathrm{U}_\gamma\left(\hat{\gamma}^0\right)^T K_{\gamma\gamma}^{-1}(\hat{\gamma}^0) \mathrm{U}_\gamma\left(\hat{\gamma}^0\right) \\
=& \frac{\left\{\sum_{i=1}^n z_i e^{0z_i}\left[\hat{t}_i+1+0z_i-\psi(e^{0z_i})\right]\right\}^T \left\{\sum_{i=1}^n z_i e^{0z_i}\left[\hat{t}_i+1+0z_i-\psi(e^{0z_i})\right]\right\}}{\sum_{i=1}^n z_i^2 e^{0z_i}[e^{0z_i}\psi'(e^{0z_i})-1]} \\
=& \frac{\left\{\sum_{i=1}^n z_i\left[\hat{t}_i+1-\psi(1)\right]\right\}^T \left\{\sum_{i=1}^n z_i\left[\hat{t}_i+1-\psi(1)\right]\right\}}{\sum_{i=1}^n z_i^2[\psi'(1)-1]} \\
=& \frac{\left\{\sum_{i=1}^n z_i\left[\hat{t}_i+1-\psi(1)\right]\right\}^2}{\sum_{i=1}^n z_i^2[\psi'(1)-1]}
\end{aligned}
$$

$$
\xi_{SR} \underset{n\to\infty}{\overset{H_0}{\sim}} \chi^2_{(1)}.
$$

$$
t_i = -\frac{y_i}{\mu} - \log(\mu) + \log(\phi_i y_i)
$$

4. Supor $Y_i \overset{\text{ind}}{\sim} \mathrm{Q}(y_i, \mu_i)$, em que $\mathrm{E}(Y_i) = \mu_i$ e $\mathrm{Var}(Y_i) = \sigma^2\mu_i(1-\mu_i)$ para $0 < y_i, \mu_i < 1$, $i = 1, \ldots, n$, com parte sistemática dada por $\log\{\mu_i/(1-\mu_i)\} = 1 + \beta x_i$. Obter a quase-escore $\mathrm{U}_\beta$ e a quase-Fisher $\mathrm{K}_{\beta\beta}$. Mostre que a estatística do teste quase-escore para testar $\mathrm{H}_0 : \beta = 0$ contra $\mathrm{H}_1 : \beta \neq 0$ pode ser expressa na forma

$$\xi_{SR} = \frac{1}{e\hat{\sigma}_0^2}\frac{[\sum_{i=1}^n x_i\{y_i(1+e) - e\}]^2}{\sum_{i=1}^n x_i^2}.$$

Apresente a estimativa $\hat{\sigma}_0^2$. Qual a distribuição nula assintótica da estatística do teste?

**Resolução**

Segue que $\mathrm{Var}(Y_i) = \sigma^2 V(\mu_i) \implies V(\mu_i) = \mu_i(1-\mu_i)$. Também, $\log\{\mu_i/(1-\mu_i)\} = 1 + \beta x_i \implies \mu_i = \frac{\exp\{1+\beta x_i\}}{1+\exp\{1+\beta x_i\}}$. A quase-escore pode ser obtida a partir de

$$\mathrm{U}_\beta = \sum_{i=1}^n \frac{(y_i - \mu_i)}{\sigma^2 V(\mu_i)} D_i,$$

em que

$$D_i = \frac{\partial \mu_i}{\partial \beta} = \frac{x_i \exp\{1+\beta x_i\}(1+\exp\{1+\beta x_i\}) - x_i \exp\{1+\beta x_i\}\exp\{1+\beta x_i\}}{[1+\exp\{1+\beta x_i\}]^2}$$

$$D_i = \frac{x_i \exp\{1+\beta x_i\}}{[1+\exp\{1+\beta x_i\}]^2} = x_i \frac{\exp\{1+\beta x_i\}}{1+\exp\{1+\beta x_i\}}\frac{1}{1+\exp\{1+\beta x_i\}} = x_i\mu_i(1-\mu_i).$$

Assim,

$$\mathrm{U}_\beta = \sum_{i=1}^n \frac{(y_i - \mu_i)}{\sigma^2 \cancel{\mu_i(1-\mu_i)}} x_i \cancel{\mu_i(1-\mu_i)} = \sum_{i=1}^n \frac{x_i(y_i - \mu_i)}{\sigma^2}.$$

A quase-Fisher pode ser obtida a partir de

$$\mathrm{K}_{\beta\beta} = \sum_{i=1}^n \frac{D_i^2}{\sigma^2 V(\mu_i)} = \frac{[x_i\mu_i(1-\mu_i)]^2}{\sigma^2\mu_i(1-\mu_i)} = \sum_{i=1}^n \frac{x_i^2\mu_i(1-\mu_i)}{\sigma^2}.$$

Para testar $\mathrm{H}_0 : \beta = \beta_0 = 0$ contra $\mathrm{H}_1 : \beta \neq 0$, a estatística do teste quase-escore é dada por

$$\xi_{SR} = \frac{\mathrm{U}_\beta(\beta_0)^2}{\mathrm{K}_{\beta\beta}(\beta_0)}.$$

Sob $H_0$, segue que $\mu_{i0} = \frac{\exp\{1\}}{1+\exp\{1\}} = \frac{e}{1+e}$. Logo,

$$\mathrm{U}_\beta(\beta_0) = \frac{1}{\hat{\sigma}_0^2}\sum_{i=1}^n x_i\left(y_i - \frac{e}{1+e}\right) = \frac{1}{\hat{\sigma}_0^2}\sum_{i=1}^n x_i\left(\frac{y_i(1+e) - e}{1+e}\right)$$

$$\implies \mathrm{U}_\beta(\beta_0)^2 = \frac{1}{(\hat{\sigma}_0^2)^2}\left[\sum_{i=1}^n x_i\left(\frac{y_i(1+e)-e}{1+e}\right)\right]^2 = \frac{1}{(\hat{\sigma}_0^2)^2}\left[\sum_{i=1}^n x_i\left\{y_i(1+e) - e\right\}\right]^2\left(\frac{1}{1+e}\right)^2.$$

Também,

$$\mathrm{K}_{\beta\beta}(\beta_0) = \frac{1}{\hat{\sigma}_0^2}\sum_{i=1}^n x_i^2\left(\frac{e}{1+e}\right)\left(\frac{1}{1+e}\right) = \frac{1}{\hat{\sigma}_0^2}\sum_{i=1}^n x_i^2\frac{e}{(1+e)^2},$$

logo

$$\xi_{SR} = \frac{1}{(\hat{\sigma}_0^2)^{\cancel{2}}}\left[\sum_{i=1}^n x_i\left\{y_i(1+e) - e\right\}\right]^2\cancel{\left(\frac{1}{1+e}\right)^2}\cancel{\hat{\sigma}_0^2}\frac{1}{\sum_{i=1}^n x_i^2}\frac{\cancel{(1+e)^2}}{e}$$

$$\xi_{SR} = \frac{1}{e(\hat{\sigma}_0^2)} \frac{\left[\sum_{i=1}^{n} x_i \left\{y_i(1+e) - e\right\}\right]^2}{\sum_{i=1}^{n} x_i^2}.$$

$$\xi_{SR} \underset{n\to\infty}{\overset{H_0}{\sim}} \chi^2_{(1)}$$

Um estimador de momentos para $\sigma^2$ é dado por

$$\hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^{n} \frac{(y_i - \hat{\mu}_i)^2}{V(\hat{\mu}_i)} = \frac{1}{n-1} \sum_{i=1}^{n} \frac{(y_i - \hat{\mu}_i)^2}{\hat{\mu}_i(1-\hat{\mu}_i)},$$

Logo, sob $\mathrm{H}_0$, segue que

$$\hat{\sigma}_0^2 = \frac{1}{n-1} \sum_{i=1}^{n} \frac{\left[y_i - \frac{e}{1+e}\right]^2}{e} (1+e)^2 = \frac{1}{e(n-1)} \sum_{i=1}^{n} \left[y_i(1+e) - e\right]^2 \frac{\cancel{(1+e)^2}}{\cancel{(1+e)^2}}$$

$$\hat{\sigma}_0^2 = \frac{1}{e(n-1)} \sum_{i=1}^{n} \left[y_i(1+e) - e\right]^2.$$

5. Supor $Y_{ij} \sim \mathrm{Q}(y_{ij}, \mu)$, em que $\mathrm{E}(Y_{ij}) = \mu$ e $\mathrm{Var}(Y_{ij}) = \sigma^2\mu(1+\mu)$ com parte sistemática dada por $\log(\mu) = \eta = \beta$, $\mu > 0$ e $\mathrm{R}_i(\alpha)$ simétrica para $i = 1, \ldots, m$ e $j = 1, 2$. Obter $\hat{\beta}_G$ e $\mathrm{Var}(\hat{\beta}_G) = H_1^{-1}(\beta)$. Mostre que a estatística do teste de Wald para testar $\mathrm{H}_0 : \beta = 1$ contra $\mathrm{H}_1 : \beta \neq 1$ pode ser expressa na forma

$$\xi_W = \frac{2n\bar{y}\left\{\log(\bar{y}) - 1\right\}^2}{\hat{\sigma}^2(1+\bar{y})(1+\hat{\alpha})}.$$

Descreva $\hat{\sigma}^2$ e $\hat{\alpha}$. Qual a distribuição nula assintótica da estatística do teste?

**Resolução**

Segue que $\mathrm{Var}(Y_i) = \sigma^2 V(\mu) \implies V(\mu) = \mu(1+\mu)$. Também, $\log(\mu) = \beta \implies \mu = e^\beta$. A quase-escore pode ser obtida a partir de

$$\mathrm{U}_\beta = \sum_{i=1}^m \mathbf{D}_i^\top \Omega_i^{-1}(\mathbf{y}_i - \boldsymbol{\mu}_i),$$

em que

$$\mathbf{D}_i = \begin{bmatrix} D_{i1} \\ D_{i2} \end{bmatrix}, \quad D_{ij} = \frac{\partial \mu_{ij}}{\partial \beta},$$

e $\Omega_i = \sigma^2 \mathbf{V}_i^{1/2} \mathrm{R}_i(\alpha) \mathbf{V}_i^{1/2}$, em que

$$\mathbf{V}_i = \mathrm{diag}\{V(\mu_{i1}), V(\mu_{i2})\}, \quad \mathrm{R}_i(\alpha) = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}.$$

Segue que

$$\frac{\partial \mu_{ij}}{\partial \beta} = e^\beta = \mu \implies \mathbf{D}_i = \begin{bmatrix} \mu \\ \mu \end{bmatrix}$$

$$\mathbf{V}_i = \mathrm{diag}\{\mu(1+\mu), \mu(1+\mu)\} \implies \mathbf{V}_i^{1/2} = \mathrm{diag}\left\{\sqrt{\mu(1+\mu)}, \sqrt{\mu(1+\mu)}\right\}$$

$$\implies \mathbf{V}_i^{1/2}\mathrm{R}_i(\alpha) = \begin{bmatrix} \sqrt{\mu(1+\mu)} & \alpha\sqrt{\mu(1+\mu)} \\ \alpha\sqrt{\mu(1+\mu)} & \sqrt{\mu(1+\mu)} \end{bmatrix}$$

$$\implies \Omega_i = \sigma^2 \mathbf{V}_i^{1/2}\mathrm{R}_i(\alpha)\mathbf{V}_i^{1/2} = \begin{bmatrix} \mu(1+\mu) & \alpha\mu(1+\mu) \\ \alpha\mu(1+\mu) & \mu(1+\mu) \end{bmatrix}$$

$$\implies \Omega_i^{-1} = \frac{1}{\sigma^2(1-\alpha^2)\mu^2(1+\mu)^2} \begin{bmatrix} \mu(1+\mu) & -\alpha\mu(1+\mu) \\ -\alpha\mu(1+\mu) & \mu(1+\mu) \end{bmatrix}$$

$$\implies \Omega_i^{-1} = \frac{1}{\sigma^2(1-\alpha^2)\mu(1+\mu)} \begin{bmatrix} 1 & -\alpha \\ -\alpha & 1 \end{bmatrix}$$

$$\implies \mathbf{D}_i^\top \Omega_i^{-1} = \frac{1}{\sigma^2(1-\alpha^2)\mu(1+\mu)} \begin{bmatrix} \mu(1-\alpha) & \mu(1-\alpha) \end{bmatrix}$$

$$\implies \mathbf{D}_i^\top \Omega_i^{-1} = \frac{1}{\sigma^2(1+\alpha)(1+\mu)} \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

Também,

$$\mathbf{y}_i = \begin{bmatrix} y_{i1} \\ y_{i2} \end{bmatrix}, \quad \boldsymbol{\mu}_i = \begin{bmatrix} \mu \\ \mu \end{bmatrix} \implies \mathbf{y}_i - \boldsymbol{\mu}_i = \begin{bmatrix} y_{i1} - \mu \\ y_{i2} - \mu \end{bmatrix}.$$

Logo

$$\mathbf{D}_i^\top \Omega_i^{-1}(\mathbf{y}_i - \boldsymbol{\mu}_i) = \frac{y_{i1} + y_{i2} - 2\mu}{\sigma^2(1+\alpha)(1+\mu)}$$

$$\implies \mathrm{U}_\beta = \sum_{i=1}^{m} \frac{y_{i1} + y_{i2} - 2\mu}{\sigma^2(1+\alpha)(1+\mu)} = \sum_{i=1}^{m} \frac{y_{i1} + y_{i2} - 2e^\beta}{\sigma^2(1+\alpha)(1+e^\beta)}.$$

Obtém-se $\hat{\beta}_G$ a partir de

$$\mathrm{U}_\beta(\hat{\beta}_\mathrm{G}) = 0 \implies \sum_{i=1}^{m} y_{i1} + y_{i2} - 2e^{\hat{\beta}_G} = 0 \implies \hat{\beta}_G = \log\left(\frac{\bar{y}_1 + \bar{y}_2}{2}\right) = \log(\bar{y}),\ \sigma^2 \neq 0,\ \alpha \neq -1.$$

Também,

$$H_1(\beta) = \sum_{i=1}^{m} \mathbf{D}_i^\top \Omega_i^{-1} \mathbf{D}_i = \sum_{i=1}^{m} \frac{2\mu}{\sigma^2(1+\alpha)(1+\mu)} = \frac{2ne^\beta}{\sigma^2(1+\alpha)(1+e^\beta)}$$

$$\implies \mathrm{Var}(\hat{\beta}_G) = H_1^{-1}(\beta) = \frac{\sigma^2(1+\alpha)(1+e^\beta)}{2ne^\beta}.$$

Para testar $\mathrm{H}_0 : \beta = \beta_0 = 1$ contra $\mathrm{H}_1 : \beta \neq 1$, a estatística do teste de Wald é dada por

$$\xi_W = \frac{(\hat{\beta}_G - \beta_0)^2}{\mathrm{Var}(\hat{\beta}_G)} = \frac{[\log(\bar{y}) - 1]^2 2n\bar{y}}{\hat{\sigma}^2(1+\bar{y})(1+\hat{\alpha})} \underset{m\to\infty}{\overset{H_0}{\sim}} \chi^2_{(1)},$$

em que $\hat{\sigma}^2$ e $\hat{\alpha}$ são estimadores consistentes de $\sigma^2$ e $\alpha$, respectivamente. Possíveis estimadores consistentes de $\sigma^2$ e $\alpha$ são dados por

$$\hat{\sigma}^2 = \frac{1}{2m-1} \sum_{i=1}^{m} \sum_{j=1}^{2} \frac{(y_{ij} - \hat{\mu}_{ij})^2}{V(\hat{\mu}_{ij})},$$

$$\hat{\alpha} = \frac{1}{m\hat{\sigma}^2} \sum_{i=1}^{m} \frac{1}{2} \sum_{j=1}^{2} \sum_{j'=1 (j' \neq j)}^{2} \frac{(y_{ij} - \hat{\mu}_{ij})}{V(\hat{\mu}_{ij})} \frac{(y_{ij'} - \hat{\mu}_{ij'})}{V(\hat{\mu}_{ij'})} = \frac{1}{m\hat{\sigma}^2} \sum_{i=1}^{m} \frac{(y_{i1} - \hat{\mu}_{i1})}{V(\hat{\mu}_{i1})} \frac{(y_{i2} - \hat{\mu}_{i2})}{V(\hat{\mu}_{i2})}.$$